1. Intro to particle physics
2. Particles, Fields & Symmetry
   Symmetry → $\underset{Colour}{Su(3)} \times \underset{electro}{Su(2)} \times \underset{weak}{U(1)}$ (electro - weak)
   Particles, Fields: Quantum Field Theory (QTF)
3. Electroweak Interactions
   Leptons ($\nu_{e,\mu,\tau}, e, \mu, \tau, W^{\pm}, Z$)
   Quarks ($u, d, c, s, t, b, \dots, W^{\pm}, Z$)
4. Higgs mechanism
   Generates mass

Wiley- Manchester
Particle Physics; Martin & Shaw

2: Particles, Fields & Symmetries

28 January 2013 10:44

2.1 Action Principle & Quantum Mechanics

Action principle is a central idea in classical mechanics which carries over into quantum mechanics and quantum field theory.

Consider the simplest 1 dimensional dynamical system

Particle with position x(t)

Action

$$S = \int dt\, L$$
$$= \int dt\, (K - V)$$
$$= \int dt \left(\frac{1}{2} m\dot{x}^2 - V(x)\right)$$

Where $\dot{x} = \frac{d}{dt}x$

To find equations of motion, we have to find the extrema of the action⇒ Lagrange's equations.

Eqs of motion follow by "minimising" (actually "extremising") the action S

Vary trajectory from x(t) to x(t)+a(x)

Since

$$S = \int dt\, L(x, \dot{x})$$

Variation of action

$$\delta S = \int dt \left(\delta x \frac{\delta L}{\delta x} + \delta \dot{x} \frac{\delta L}{\delta \dot{x}}\right)$$
$$= \int dt \left(\delta x \frac{\delta L}{\delta x} + \frac{d}{dt}(\delta x)\frac{\delta L}{\delta \dot{x}}\right)$$
$$= \int dt \delta x \left(\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}\right)$$

Require $\delta S = 0$ for any δx

$$\Rightarrow \boxed{\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}} = 0}$$

Since

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - V(x)$$
$$\Rightarrow \frac{\delta L}{\delta x} = -\frac{dV}{dx}$$
$$\frac{\delta L}{\delta \dot{x}} = m\dot{x}$$

So Lagrange Eq is

$$-\frac{dV}{dx} - \frac{d}{dt}(m\dot{x}) = 0$$

That is

$$m\ddot{x} = -\frac{dV}{dx} = F$$

=force

This is just Newton's 2nd law.

Symmetries + Noether's Theorem

Consider a transformation

$x \to x + a(x)$

Change in Lagrangian is

$$\delta L = a\frac{\delta L}{\delta x} + \frac{d}{dt}(a)\, \delta L/\delta \dot{x}$$
$$= a\frac{\delta L}{\delta x} - a\frac{d}{dt}\frac{\delta L}{\delta \dot{x}} + \frac{d}{dt}\left(a\frac{\delta L}{\delta \dot{x}}\right)$$

$$= a\left(\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}\right) + \frac{d}{dt}\left(a\frac{\delta L}{\delta \dot{x}}\right)$$

So if $\delta L = 0$ then using the eq of motion we find

$$\frac{d}{dt}\left(a\frac{\delta L}{\delta \dot{x}}\right) = 0$$

$$\Rightarrow J = a(x)\, \delta L/\delta \dot{x}$$

Is a conserved quantity

Noether's theorem:-

Each symmetry of the Lagrangian (transformations with $\delta L = 0$) corresponds to a conserved quantity J. And vice-versa!

e.g. translations

Transformation is $x \to x + a$ where a is constant

⇒conserved quantity is

$$J = a\frac{\delta L}{\delta \dot{x}} = am\dot{x}$$

That is momentum $p = m\dot{x}$

Is conserved, i.e.

$$\frac{dp}{dt} = 0$$

Similarly for rotations ⇔conservation of angular momentum

Quantum Mechanics

Recall some basic QM, using harmonic oscillator as an example

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

$$v \sim x^2 \Rightarrow F = m\omega^2 x$$

$$p = \frac{\delta L}{\delta \dot{x}} = m\dot{x}$$

So

$$H(x, p) = \frac{p^2}{2m} + V(x)$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

In QM, the dynamical variables x and p become operators $\hat{x}$ and $\hat{p}$ act on states $|\square\rangle$

These operators can have non-trivial commutation relations depending on $\hbar$

$$[\hat{x}, \hat{p}] = i\hbar$$

To solve the HO, we introduce raising and lowering operators (soon to be called "creation" and "annihilation" operators)

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})$$

With the normalisation, the CRs for $\hat{x}$ *and* $\hat{p}$ become

$$[\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)$$

CRs

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Rightarrow [\hat{a}, \hat{a}^\dagger] = 1$$

Define energy eigenstates $|\square\rangle$

Ground state $|0\rangle$

Defined by $\hat{a}|0\rangle = 0$

Then $|1\rangle = \hat{a}^{\dagger}|0\rangle$

$|n+1\rangle = \hat{a}^{\dagger}|n\rangle$

Numbre operator

$\widehat{N} = \hat{a}^{\dagger}\hat{a}$

$\Rightarrow \widehat{N}|n\rangle = n|n\rangle$

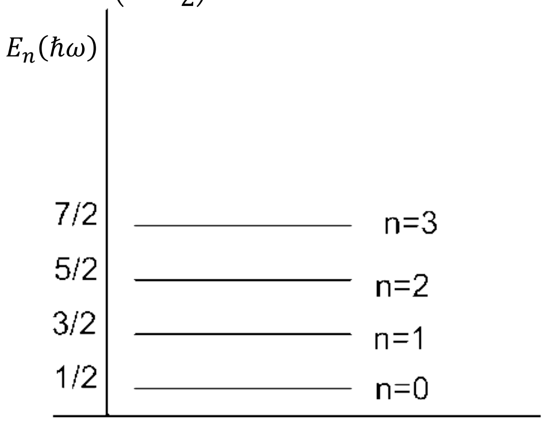
Energy eigenstates are eigenstates of $\widehat{H}$

$\widehat{H}|n\rangle = E_n|n\rangle$

E_n =energy eigenvalue

We can show

$$\widehat{H} = \left(\widehat{N} + \frac{1}{2}\right)\hbar\omega$$



Alternative QM

Feynman path integral

Transition probability to go from point $x(0)$ to $x(t)$ is given by

$$\langle\, x(t) \mid x(0)\,\rangle = \int_{x(0)}^{x(t)} \mathcal{D}x(t) e^{\frac{i}{\hbar}S}$$

$$\langle\, x(t) \mid x(0)\,\rangle \to Prob = |\square|^2$$

$$\int_{x(0)}^{x(t)} \mathcal{D}x(t) e^{\frac{i}{\hbar}S} = integral\ over\ paths$$

$$S = \int dt\, L(x, \dot{x})$$

i.e. integrate over all paths weighted by a phase factor given by the action for that path.

Notice that in classical dynamics where $\hbar \to 0$, the path with minimum (extremum) action S that dominates.

["Method of stationary phase"]

$\Leftrightarrow$action principle of classical mechanics

2.2 Quantum fields

Start with electromagnetism. Classically, described electric and magnetic fields $\underline{E}$ and $\underline{B}$ or alternatively by "potentials"

$\underline{E} = \underline{\nabla}\phi$

$\underline{B} = \underline{\nabla} \times \underline{A}$

From now on, call $\phi, \underline{A}$ the electromagnetic fields

Electromagnetism is already a fully special relativistic theory

We can write fields in covariant, 4-vector, notation :-

$$A_\mu = \left(\frac{1}{c}\phi, \underline{A}\right)$$

$\mu = 0,1,2,3$

In QM, electromagnetism is described by photons ("quanta of the e/m field")

Photons are neutral, spin 1.

There is a unifying description in terms of quantum fields

$A_\mu \to \hat{A}_\mu$, an operator with non-trivial commutation relations.

States $|\square\rangle$ are eigenstates of $\hat{A}_\mu$, these are photons

In fact, the electromagnetic field is complicated to quantise. We start with a sim[;er case
Rule

Type of field	Spin of particle	"Statistics"
4-vector	1	Bose-Einstein
Spinor	1/2	Fermi-Dirac
Scalar	0	Bose-Einstein
Tensor (metric $g_{\mu\nu}$)	2 (graviton)	B-E

Dictionary

Non-rel particle QM	Rel QFT (Scalar field ϕ)
$x(t)$	$\phi(x^\mu)$
t	$x^\mu = (t, \underline{x})$
$\frac{d}{dt}$	$\frac{\delta}{\delta x^\mu} \equiv \delta_\mu$
$\int dt$	$\int d^4x$
$L(x, \dot{x})$	$L(\phi, \delta_\mu \phi)$

$$S = \int dt\, L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2$$

The Lagrangian for a single real scalar field is

$$S = \int d^4x\, L(\phi, \delta_\mu \phi)$$

$$= \int d^4x \left(g^{\mu\nu} \delta_\mu \phi \delta_\nu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

Where

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\Rightarrow g^{\mu\nu} \delta_\mu \phi \delta_\nu \phi = -\frac{\delta\phi}{\delta t}\frac{\delta\phi}{\delta t} + \frac{\delta\phi}{\delta x}\frac{\delta\phi}{\delta x} + \frac{\delta\phi}{\delta y}\frac{\delta\phi}{\delta y} + \frac{\delta\phi}{\delta z}\frac{\delta\phi}{\delta z}$$

Eq of motion for field ϕ is derived using Lagrange's eqs.

Action principle,

$$O = \delta S = \int d^4x \left(\delta\phi \frac{\delta L}{\delta \phi} + \delta(\delta_\mu \phi) \frac{\delta L}{\delta(\delta_\mu \phi)} \right)$$

$$\delta(\delta_\mu \phi) \to \delta_\mu(\delta\phi)$$

Integrate by parts

$$= \int d^4x\, \delta\phi \left(\frac{\delta L}{\delta\phi} - \delta_\mu \left(\frac{\delta L}{\delta(\delta_\mu \phi)} \right) \right)$$

⇒Eq of motion is

$$\frac{\delta L}{\delta \phi} - \delta_\mu \frac{\delta L}{\delta(\delta_\mu \phi)} = 0$$

If

$$L = \frac{1}{2} \left(g^{\mu\nu} \delta_\mu \phi \delta_\nu \phi - m^2 \phi^2 \right)$$

$$\Rightarrow \frac{\delta L}{\delta \phi} = -m\phi$$

$$\frac{\delta L}{\delta(\delta_\mu \phi)} = g^{\mu\nu} \delta_\nu \phi \Rightarrow \frac{\delta L}{\delta(\delta_\mu \phi)} = g^{\mu\nu} \delta_\mu \delta_\nu \phi$$

Eq of motion is

$$g^{\mu\nu} \delta_\mu \delta_\nu \phi + m^2 \phi = 0$$

$$g^{\mu\nu} \delta_\mu \delta_\nu = \delta^2$$

$$\Leftrightarrow (\delta^2 + m^2)\phi = 0$$

$$\Leftrightarrow \boxed{\left(-\frac{\delta^2}{\delta t^2} + \nabla^2 + m^2\right)\phi = 0}$$

Klein-Gordon equation

Relativistic field eq

Solutions

The KG eq has simple plane-wave solutions (Just like Maxwell's eqs)

Soln is

$\phi(t, \underline{x}) = e^{i(-Et + \underline{p}.\underline{x})}$

Clear that this is a solution if

$E^2 - |\underline{p}|^2 + m^2 = 0$

Why

$\left(-\frac{\delta^2}{\delta t^2} + \nabla^2\right)\phi_{SOLN} = -E^2 + |\underline{p}|^2$

So the plane-wave soln is valid provided E and $\underline{p}$ are interpreted as energy and momentum and satisfy the relativistic energy-momentum mass eq

Nb: using units C=1 everywhere and $\hbar = 1$ for QM

Quantum fields

We now need to quantise this field analogy with QM and harmonic oscillator

QM $x \to \hat{x}$ acts on states $|\square\rangle$

SHO convenient

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$

QFT let $\phi \to$ Operator $\widehat{\phi}$ acting on states $|\square\rangle$

$\widehat{\phi}(x) = \int d^4 p e^{ipx}\hat{a}(p) + e^{ipx}\hat{a}^\dagger(p)$

Where

$e^{-ipx} \equiv e^{-ig^{\mu\nu}p_\mu x_\nu}$

$= e^{i(Et - \underline{p}.\underline{x})}$

This operator satisfies the equation of motion

Define states

Vacuum states $|0\rangle$ defined by $\hat{a}|0\rangle = 0$

$\hat{a}$ is an annihilation operator

$\hat{a}^\dagger$ is a creaton operator

$\hat{a}^\dagger(p)|0\rangle = |p\rangle$

$\to$ single particle states with 4-mom $p^\mu = (E, \underline{p})$

Carry on

$\hat{a}^\dagger(q)|p\rangle = |p, q\rangle$

=2 particle state

The $\hat{a}, \hat{a}^\dagger$ have Commutation Relations

$[\hat{a}(p), \hat{a}^\dagger(q)] = \delta^{(4)}(p, q)$

Only $\neq 0$ if $p^\mu = q^\mu$

Dirac delta fn

Conclude

The states of a quantised field are particles !!! (sic)

e.g. electromagnetic field $\leftrightarrow$ photons

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Symmetry and noether current

Action has symmetry where

$\phi \to e^{i\alpha}\phi$ $(= 1 + i\alpha + O(\alpha))\phi)$

$\phi^* \to e^{-i\alpha}\phi^*$

if $\alpha =$ constant parameter

Noether's theorem implies there is a corresponding conservatiion law

Noether's Theorem:
~~~

$$J_\mu = \delta\phi \frac{\delta L}{\delta(\delta_\mu \phi)} + \delta\phi^* \frac{\delta L}{\delta(\delta_\mu \phi^*)}$$

Where

$$\delta\phi = i\alpha\phi$$

Where α is infintesimal

$$\delta\phi^* = -i\alpha\phi^*$$

$$\Rightarrow J_\mu = i\alpha(\phi\delta_\mu\phi^* - \phi^*\delta_\mu\phi)$$

Noeher's theorem says this is conserved

$$\delta^\mu J_\mu = 0$$

Since this holds for <u>any</u> α, we can just take

$$J_\mu = i(\phi\delta_\mu\phi^* - \phi^*\delta_\mu\phi)$$

This is conserved <u>Current</u>

Recall from electromagnetism that current density $\underline{j}$ and charge density ρ make up a 4-vector $J_\mu = (\rho, \underline{j})$

[units c=1]

Conservation of electric charge

$$\Rightarrow \underline{\nabla}.\underline{j} = \frac{\delta\rho}{\delta t}$$

$$\Rightarrow -\frac{\delta\rho}{\delta t} + \underline{\nabla}.\underline{j} = 0$$

This is just

$$\delta^\mu J_\mu = 0$$

$$\delta_\mu = \left(\frac{\delta}{\delta t}, \underline{\nabla}\right)$$

Here can define the <u>charge</u>

$$Q = \int d^3x J^0$$

$$\Rightarrow \frac{d}{dt} Q = 0$$

$$\left(\text{Why } \frac{dQ}{dt} = \int d^3x(\delta^0 J_0 + \underline{\nabla}\underline{J}), \qquad \text{so } \delta^\mu J_\mu = 0 \Rightarrow \frac{dQ}{dt} = 0\right)$$

So we find that for a complex scalar field the symmetry

$$\phi \to e^{i\alpha}\phi$$

$$\phi^* \to e^{-i\alpha}\phi^*$$

Implies a conserved current

$$J_\mu = i(\phi\delta_\mu\phi^* - \phi^*\delta_\mu\phi)$$

Satisfying

$$\delta^\mu J_\mu = 0$$

[exercise: Check using eqs of motion]

And a conserved charge

$$Q = \int d^3x J_0$$

satisfying

$$\frac{dQ}{dt} = 0$$

This is the first example of a <u>gauge symmetry</u>

The "charge" can be interpreted as electric charge so $\boxed{\text{electric charge} \leftrightarrow \text{gauge symmetry}}$

Write symmetry as $\phi \to U\phi$

Where $U = e^{i\alpha}$

i.e. with $U^*U = 1$

This is called U(1) symmetry

If we had N scalar fields

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

The symmetry would be $\phi \to U\phi$ where U is an NxN matrix satisfying $U^\dagger U = 1$

$\left(U^\dagger = U^{T*}\right)$

I.e. U is a unitary matrix

In this case, the symmetry is U(N)

Quantised charged scalar field.

Here we write the quantum filed in terms of quantum annihilation operators,

$$\widehat{\phi}(x) = \int d^4p \left(e^{-ipx}\hat{a}(p) + e^{ipx}\hat{b}^\dagger(p)\right)$$

$$\widehat{\phi}^*(x) = \int d^4p \left(e^{-ipx}\hat{b}(p) + e^{ipx}\hat{a}^\dagger(p)\right)$$

i.e. we have two types of particle created by

$\hat{a}^\dagger$ and $\hat{b}^\dagger$

Exercise (Hard- uses Fourier transforms and delta functions)

From the definitions of J_μ and Q

We can show

$$Q = \int d^4p \left(\hat{a}^\dagger(p)\hat{a}(p) - \hat{b}^\dagger(p)\hat{b}(p)\right)$$

$$= \int d^4p\left(N_+(p) - N_-(p)\right)$$

$$= N_+ - N_-$$

So we interpret $\hat{a}^\dagger|0\rangle$ as creating a +ve charge particle and $\hat{b}^\dagger|0\rangle$ as creating a -ve charge particle

Dirac Field and Spin 1/2 Particles

To describe spin 1/2 particles, we need a new type of field e.g. spin 1/2 electron can have 2 spin states, $s_z = \pm\hbar/2$ (Spin "up" or "down")

In particle physics, choose to measure components of spin along the direction of motion

Define <u>helicity</u>

$$\lambda\hbar = \frac{\underline{s}.\underline{p}}{|\underline{p}|}$$

Where $\underline{p}$ = 3-momentum

So

$$\lambda = \pm\frac{1}{2}$$

Jargon,

$\lambda = +\frac{1}{2}$ is called left-handed

$\lambda = -\frac{1}{2}$ is called right-handed

So the 2 electron spin states are called e_L, e_R

Because every particle has an anti-particle of opposite charge, we also have positrons e_R^+, e_L^+

→ Field describing electron/positron has 4 components

Dirac field

$$\psi = \begin{pmatrix} \psi_L^- \\ \psi_L^+ \\ \psi_R^- \\ \psi_L^- \end{pmatrix}$$

Like a vector, but not in space. Called a <u>SPINOR</u>!!eleven!1

Dirac Lagrangian,

$$S = \int d^4x L = \int d^4x \overline{\psi}\left(i\gamma^\mu\delta_\mu + m\right)\psi$$

Where

$$\overline{\psi} \sim \psi^\dagger = \psi^{T*}$$

So $\bar{\psi}$ is a 1x4 vector and ψ is a 4x1 vector

The γ^μ are a set of 4x4 matrices for μ=0,1,2,3

$$\gamma^\mu \delta_\mu = -\gamma^0 \frac{\delta}{\delta t} + \gamma^1 \frac{\delta}{\delta t} + \gamma^2 \frac{\delta}{\delta t} + \gamma^3 \frac{\delta}{\delta t}$$

→ each 4x4 matrices

Note: this is 1st order in derivatives!

Equation of motion :-

Use lagrange eq wih $\bar{\psi}$:-

$$\frac{\delta L}{\delta \bar{\psi}} - \delta_\mu \frac{\delta L}{\delta(\delta_\mu \bar{\psi})} = 0$$

$$\rightarrow (\delta_\mu \bar{\psi}) = 0$$

$$\rightarrow \boxed{(i\gamma^\mu \delta_\mu + m)\psi = 0}$$

Dirac Eq

Noether's theorem:-

Just like the charged scalar field, the dirac action has a symmetry

$$\psi \rightarrow e^{i\alpha}, \qquad \bar{\psi} \rightarrow e^{-i\alpha}\bar{\psi}$$

With α= constant

So there is a conserved current

$$J^\mu = \delta\bar{\psi} \frac{\delta L}{\delta(\delta_\mu \bar{\psi})} + \frac{\delta L}{\delta(\delta_\mu \psi)} \delta\psi$$

n.b. order is important since $\delta\psi = i\alpha\psi$

$$= (\bar{\psi} i\gamma^\mu)(i\alpha\psi) = -\alpha\bar{\psi}\gamma^\mu\psi$$

Since this is conserved for any α, we just take the conserved current to be

$$J^\mu = \bar{\psi}\gamma^\mu\psi$$

Conserved $\delta_\mu J^\mu = 0$

J^μ corresponds to electric charge

History (~1929)

We want a field ψ that satisfies a relativistic wave eq

$$(-\delta^2 + m^2)\psi = 0$$

Dirac noticed that this can be derived from a simpler, 1st order eq

$$\Rightarrow \boxed{(i\gamma^\mu \delta_\mu + m)\psi = 0}$$

Postulate eq

$$(i\gamma^\mu \delta_\mu + m)\psi = 0$$

i →Real Lagrangian

γ^μ →Needed to countreract with δ_μ (relativity) for Lorentz invariance

m → single power of m for dimensions

But what are γ^μ?

What are δ_μ?

If this eq holds, then must also have

$$(-i\,\gamma^\nu \delta_\nu + m)(i\gamma^\mu \delta_\mu + m)\psi = 0$$

$$= \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\delta_\mu \delta_\nu + m^2\right)\gamma$$

$\{A, B\}$ = Anticommutator = $AB + BA$

Require

$$= (-g^{\mu\nu}\delta_\mu \delta_\nu + m^2)\psi$$

The γ^μ <u>must</u> satisfy

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}1$$

This can only be satisfied if he γ^μ are 4x4 matrices!

But ψ was meant to describe a relativistic electron

→ so why 4-components?

Conclude (1) spin 1/2, so e_L^- and e_R^-

(2) anti-particles, positron e_L^+ and e_R^+

Prediction of antimatter

Electromagnetic Field

Maxwell's equations

$$\begin{matrix}(1)\\(2)\end{matrix}\begin{cases}\underline{\nabla}.\underline{E}=\frac{1}{\varepsilon_0}\rho\\ \underline{\nabla}\times\underline{B}-\mu_0\varepsilon_0\frac{\delta\underline{E}}{\delta t}=\mu_0\underline{j}\end{cases}$$

$$\begin{matrix}(3)\\(4)\end{matrix}\begin{cases}\underline{\nabla}.\underline{B}=0\\ \underline{\nabla}\times\underline{E}+\frac{\delta\underline{B}}{\delta t}=0\end{cases}$$

$$(3)+(4)\Rightarrow\underline{B}=\underline{\nabla}\times\underline{A}$$

$$\underline{E}=-\underline{\nabla}\phi-\frac{\delta\underline{A}}{\delta t}$$

Current conservation

$$\frac{\delta\rho}{\delta t}+\underline{\nabla}.\underline{j}=0$$

Then (1) + (2) become

$$\frac{1}{c^2}\frac{\delta^2\phi}{\delta t^2}-\nabla^2\phi=\frac{1}{\varepsilon_0}\rho$$

$$\frac{1}{c^2}\frac{\delta^2\underline{A}}{\delta t^2}-\nabla^2\underline{A}=-\mu_0\underline{j}$$

Where

$$C=\frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Relativistic notation

4-vector electromagnetic field $A_\mu=\left(\frac{1}{c}\phi,\underline{A}\right)$

4-vector current $J_\mu=\left(c\rho,\underline{j}\right)$

Derivatives $\delta_\mu\equiv\frac{\delta}{\delta x^\mu}=\left(\frac{1}{c}\frac{\delta}{\delta t},\underline{\nabla}\right)$

$$\Rightarrow\boxed{\delta^2A_\mu=-\mu_0J_\mu}$$

This equation of motion can be derived from a lagrangian

$$S=\int d^4x\,L=\boxed{\int d^4x\left(\frac{1}{2}A^\mu\delta^2A_\mu+A^\mu J_\mu\right)}$$

Lagrange equation →equation of motion

No mass term

⇒particles corresponding to the EM field are massless. These are the photons

NB A_μ is a real 4-vector ⇒ photons are neutral

So maxwell's electromagnetism interpreted as a quantum field theory ⇒ photons are massless, neutral, spin 1.

Interactions

The interaction of the photon field with a current is given by

$$L_{int}=A^\mu J_\mu$$

So e.g. the interaction of photons with electrons is

$$L_{int}=A^\mu\bar{\psi}\gamma_\mu\psi$$

Using the form of the current in Dirac theory

For a charged spin 0 particle

$$L_{int}=A^\mu\left(\phi\delta_\mu\phi^*-\phi^*\delta_\mu\phi\right)$$

Using J_μ for complex scalar field

Conclude

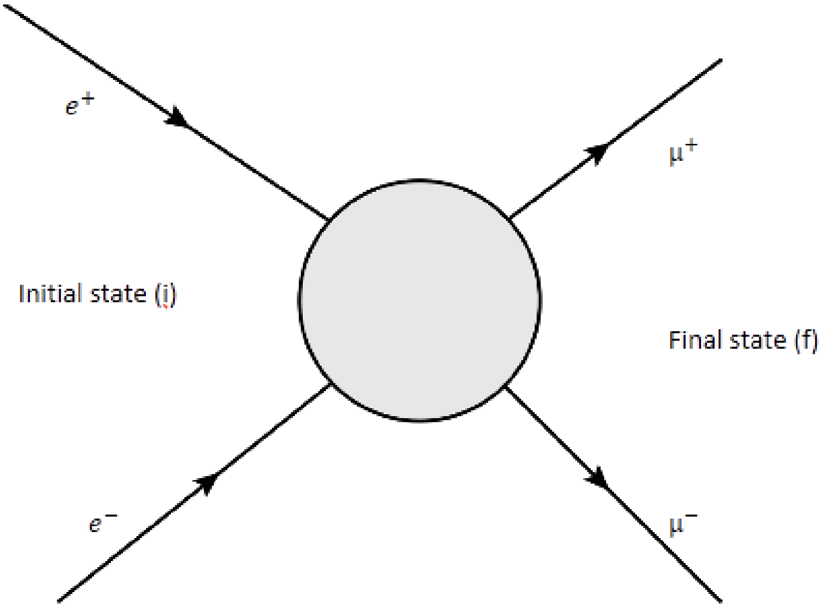
In QFT, interactions of particles are controlled by coupling of the field A_μ to the current J_μ

⇒interaction is determined by the form of the current, but current is determined by the symmetry by Noether's theorem, e.g. $\psi\to e^{i\alpha}\psi$

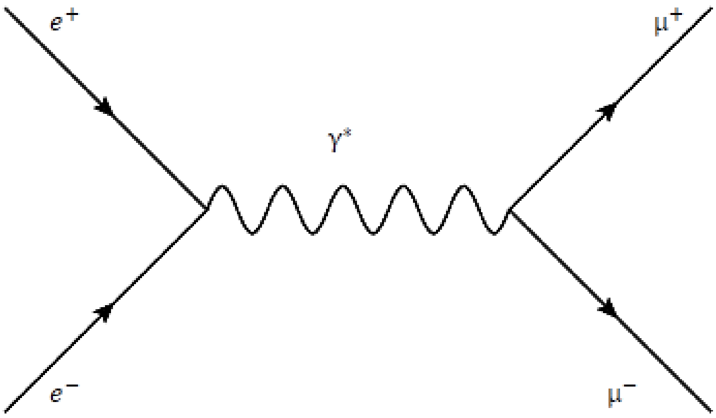
⇒ $\boxed{\text{Gauge symmetries determine all the particle interactions!}}$

2.3 Feynman Diagrams

A central problem in particle physics is to describe scattering



We need the probability of a given outcome
$Prob_{i\to f} = |\langle f \mid i \rangle|^2$
(QM!)
Cross-section=(incoming flux)
$\times\, Prob_{i\to f}$
$\times\, (final\ state\ factors)$
Feynman diagrams are pictorial representation of the scattering amplitude $\langle f \mid i \rangle$ (NB. Not probability)
For example



Feynman Rules and QED

The Quantum Electrodynamics (QED) action is

$$\int d^4x \left(\frac{1}{2} A^\mu \delta^2 A_\mu + \overline{\psi}\left(i\gamma^\mu \delta_\mu + m\right)\psi + eA^\mu \overline{\psi}\gamma_\mu \psi\right)$$

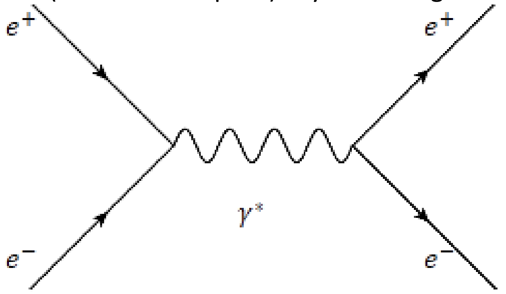
A feynman diagram describes a scattering amplitude according to rules derived from the action
Feynman diagram has 3 parts

1) External lines
   Denote wavefunctions for the "in" and "out" states
2) Internal lines
   Denote "propagators" determine how virtual particles are transmitted. Propagators read off from action
   Obtain propagators as inverse of the Fourier transform of the quadratic (non-interacting) terms in action ⇒ photon propagator $1/p^2$, $p^\mu =$ momentum
   Electron propagator
   $$\frac{i\gamma^\mu p_\mu + m}{p^2 + m^2}$$
3) Vertices

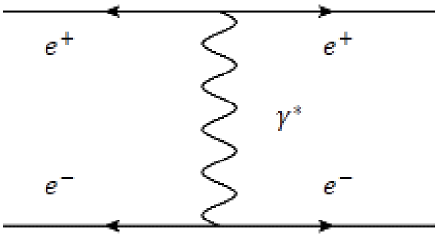
Describe the interactions for a momentum-space Feynman diagram, impose 4-momentum conservation.
Read off strength of interaction from $eA\bar{\psi}\Psi$ term. Here, e=coupling constant
In QED, this is a $\gamma e^+ e^-$ coupling

Virtual particles & resonance

In a (momentum-space) Feynman diagram like



or

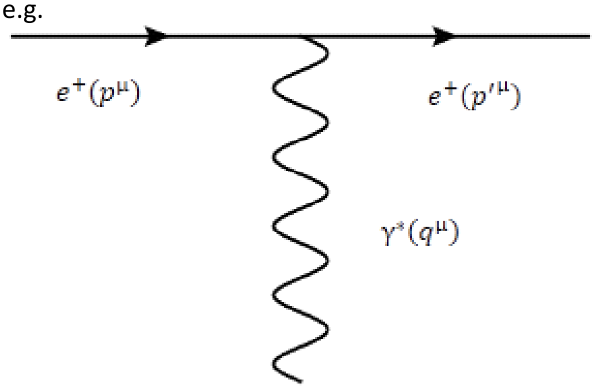


The photon γ^* does NOT satisfy the energy-momentum-mass relation of special relativity, so internal lines do not represent real particles- call them virtual

<u>Proof</u>

e.g. LEP, the γ^* would have $|\underline{p}| = 0$ and $E = 2E_{beam}$ so $E \neq c|\underline{p}|$

e.g.



4-momentum

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right)$$
$$p'^\mu = \left(\frac{E'}{c}, \underline{p}'\right)$$
$$q^\mu = \left(\frac{E_\gamma}{c}, \underline{q}\right)$$

Where

$$E_\gamma = E - E'$$
$$\underline{q} = \underline{p} - \underline{p}'$$

External e^+ are real, so satisfy

$$E^2 - c^2|\underline{p}|^2 = m^2c^4$$
$$E'^2 - c^2|\underline{p}'|^2 = m^2c^4$$

But then

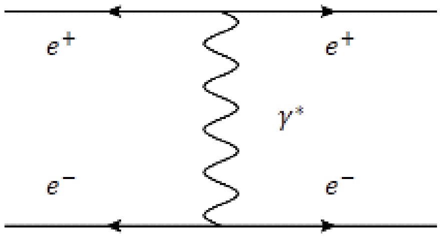
$$E_\gamma^2 - c^2|\underline{q}|^2 = (E - E')^2 - c^2(\underline{p} - \underline{p}')^2 = E^2 - c^2|\underline{p}|^2 + E'^2c^2|\underline{p}'|^2 - 2EE' + 2c^2|\underline{p}||\underline{p}'|\cos\theta$$
$$= 2m^2c^4 - 2EE' + 2c^2|\underline{p}||\underline{p}'|\cos\theta$$

To show this is non-zero, note that $E_\gamma^2 - c^2|\underline{q}|^2$ is Lorentz invariant in any convenient frame. Choose the rest frame of initial positron i.e. $\underline{p} = 0$, $E = mc^2$

Then

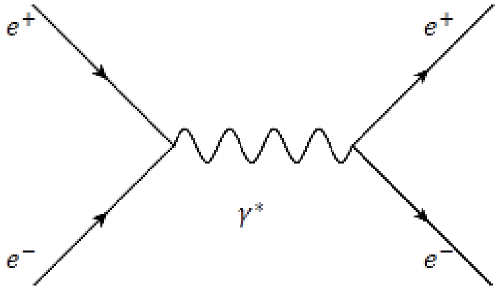
$$E_\gamma^2 - c^2|\underline{q}|^2 = 2mc^2(mc^2 - E') < 0$$

So the exchanged γ^* in Feynman diagram



has $E_\gamma^2 - c^2|\underline{q}|^2 < 0$
("Spacelike" 4-momentum)

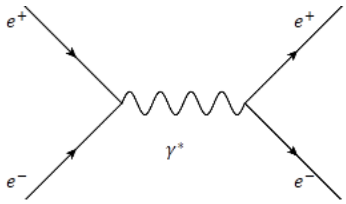
And we can show



Has $E_\gamma^2 - c^2|\underline{q}|^2 > 0$
("Timelike" 4-momentum)

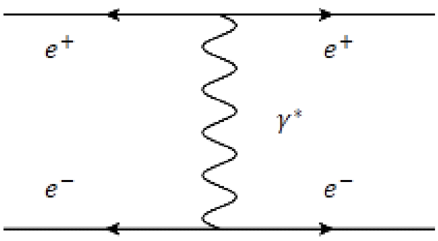
So propagators represent virtual particles which do not satisfy the energy-momentum-mass relation i.e. they are "off-shell"

Processes like



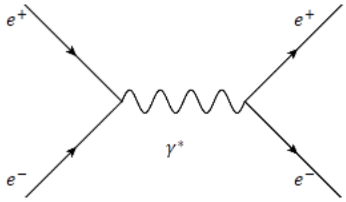
Has timelike γ^*, called "s-channel" process

Process like



Is called "t-channel" process, and γ^* is spacelike

Last time



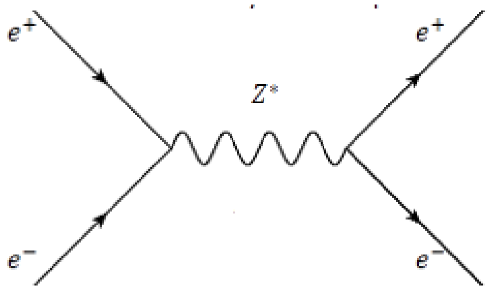
Cross section

$$\frac{1}{p^2}e^2 = \frac{1}{-E^2 + |p|^2}e^2$$

$p^2 \Rightarrow$Propagator$\Rightarrow p^2 \neq 0$ Virtual

Massive particles

As we see shortly the Z couple to e^+e^-



Cross-section

$$\sim \frac{1}{p^2 + m_Z^2} g^2$$

g^2 =coupling

$$\sim \frac{1}{-E^2 + |\underline{p}|^2 + m_Z^2} \sim \frac{1}{-E_{ch}^2 + m_Z^2}$$

This propagator becomes very big when

$$E^2 \approx |\underline{p}|^2 + m_Z^2$$

i.e. when virtual Z^* is very nearly on-shell

Big increase in cross-section when e^+e^- E_{em} is close to a particle mass "Resonance"

Another example: discovery of ψ in e^+e^- collisions at SLAC in 1974

$e^+e^- \to \psi \to e^+e^-$

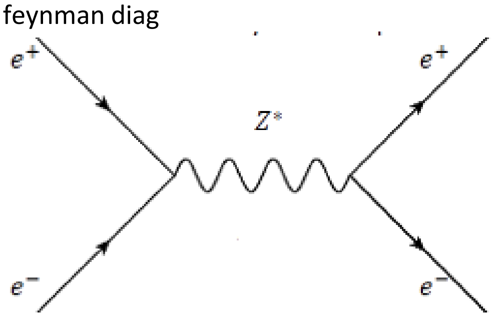
~

$m_\psi = 3.14 GeV$

Later $\psi = c\bar{c}$

Discovery of charm quark

For a particle with mass non-zero like the Z, we can have kinematics so that all of e^+, e^- and Z are on shell in the feynman diag

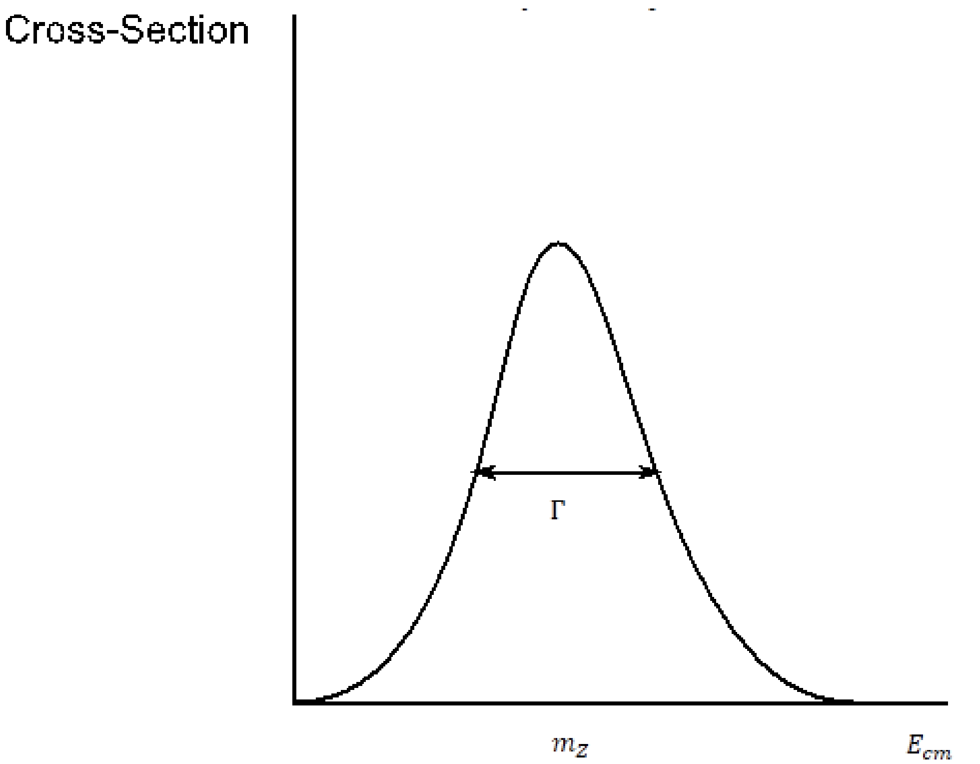


However, for a particle that can decay, the propagator is actually

$$\frac{1}{p^2 + m^2 + im\Gamma}$$

Where Γ =decay rate="width"

So width of a resonance measures the decay rate of a particle



2.4 Symmetries

So far we have used the group U(1) for QED:- $\psi \to U\psi, \quad U = e^{i\alpha}$
Every symmetry is described as a "group"
The mathematics for symmetry is called Group Theory.
Here, we need to use Lie Groups like U(1), SU(2), SU(3)...
Electroweak and strong interactions (Quantum chromodynamics) involve SU(2) and SU(3).
Hypothetical "grand unified theories" would involve bigger groups like SU(5), SO(10), E_8...

SU(2)

Suppose we have 2 flavours of quarks u,d (in addition to their colour, L or R, particle/antiparticle)
So write $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$ as a 2-cpt vector
$\Rightarrow$Action

$$S = \int d^4x(\bar{\psi}(i\gamma^\mu \delta_\mu + m)\psi)$$

$$= \int d^4x(\bar{u} \quad \bar{d})(i\gamma^\mu \delta_\mu + m)\begin{pmatrix} u \\ d \end{pmatrix}$$

$$= \int d^4x\,(\bar{u}(i\gamma^\mu \delta_\mu + m)u + \bar{d}(i\gamma^\mu \delta_\mu + m)d)$$

This has a symmetry. Lagrangeian is invariant if we let

$$\psi \to U\psi, \qquad \bar{\psi} \to \bar{\psi}U^\dagger$$

Where U is 2x2 matrix satisfying $U^\dagger U = 1$
i.e. U is a unitary matrix
So the Dirac action with 2 quark flavours has a symmetry $\psi \to U\psi$ where U is a 2x2 unitary matrix
This describes the symmetry group U(2)
This symmetry corresponds to the Lie group U(2)
Unitary 2x2 matrixes
In particle physics we are more often concerned with the group SU(2) of transformations where U=unitary, $U^\dagger U = 1$ and has $\det U = 1$
$\Rightarrow SU(2)$
"Special", i.e. det=1

<u>Group theory</u>

Any unitary matrix U can be written as $U = e^{iT}$ where $T^\dagger = T$, i.e. T =hermitian
Check

$$U^\dagger U = e^{-iT^\dagger} e^{iT} = 1$$
$$\Rightarrow T^\dagger = T$$

For the group SU(2), we write

$$U = e^{i\alpha^a T^a} \equiv e^{i(\alpha^1 T^1 + \alpha^2 T^2 + \alpha^3 T^3)}$$

Why? There are 3 unitary 2x2 matrices with det=1⇒ need 3 parameters α^a

[Any complex 2x2 unitary matrix U has 8 real parameters

Unitary $U^\dagger U$ →4 real constraints

Special det $U = 1$ →1 real constraint

Total=8-4-1=3]

U=unitary⇔ T^a =hermitian

$\det U = 1 \Leftrightarrow \mathrm{tr}\, T^a = 0$

So T^a are hermitian, traceless 2x2 matrices

Why? For any matrix A,

$\log\det A = \mathrm{tr}\log A$

$\Rightarrow \det A = e^{\mathrm{tr}\log A}$

So with $U = e^{iT}$

$\det U = e^{i\,\mathrm{tr}\, T}$

So $\det U = 1 \Leftrightarrow \mathrm{tr}\, T = 0$

For SU(2), the three T^a are traceless, 2x2 hermetian matrices. We know these- they are just the Pauli matrices σ^a

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In fact, choose $T^a = \frac{1}{2}\sigma^a$

Then we know the commutation relations

$[T^a, T^b] = i\epsilon^{abc}T^c$

Things that tell you what the commutation relation is are called "structure constants"

Where ϵ^{abc} = antisymmetric symbol

[That is $[T^1, T^2] = iT^3$ plus cyclic terms]

Knowing T^a commutation relations tells us <u>everything</u> about how to combine U transformations $\psi \to U\psi$

Essence of group theory:-

Transformations U are "group elements"

T^a are called group "generators"

α^a are "parameters"

[NB if α^a are continuous parameters, then it is a Lie group]

All the properties of the symmetry transformations are encoded in the CRs for the generators.

Mathematically, the structure described by the CRs is calld an "algebra".

This is a "Lie Algebra" of generators T^a

Note: This is precisely the structure of rotation symmetry

The T^a are just angular momentum operators.

If instead we have 3 fermions, e.g. either consider u,d,s quarks or three colours q^i, i = Red, Blue, Green

We have transformations $\psi \to U\psi$ with $U = 3x3$ unitary matrix

Group=SU(3)

Generators have CRs $[T^a, T^b] = -f^{abc}T^c$

How many generators?

$U = 3 \times 3$ complex →18 real nos

Unitary $U^\dagger U = 1$ →9 real constraints

Special $\det U = 1 \to 1$ real constraint

$\Rightarrow 18 - 9 - 1 = 8$ real parameters

So for SU(3) we have 8 generators

$T^a = 1, \dots 8$

For SU(3), the generators are the 8 traceless hermitian 3x3 matrices T^a

These are called Gell-Mann matrices λ^a

And we set $T^a = \frac{1}{2}\lambda^a$

(Look up λ^a in books)

T^a = set of 3 2x2 hermitiam matrices

Infinitesimal transformations

$U = 1 + i\alpha^a T^a + \cdots$

$\delta\psi = i\alpha^a T^a \psi$

Noether's theorem ⇒ conserved current J_μ

$$J_\mu = \delta\bar{\psi}\frac{\delta L}{\delta(\delta^\mu\bar{\psi})} + \frac{\delta L}{\delta(\delta^\mu\psi)}\delta\psi$$

$$\quad\quad\rightarrow 0, \quad\quad \rightarrow \bar{\psi}i\delta_\mu$$

$$= -\alpha^a\bar{\psi}\delta_\mu T^a\psi$$

Since this holds for any parameters α^a, there are 3 conserved currents

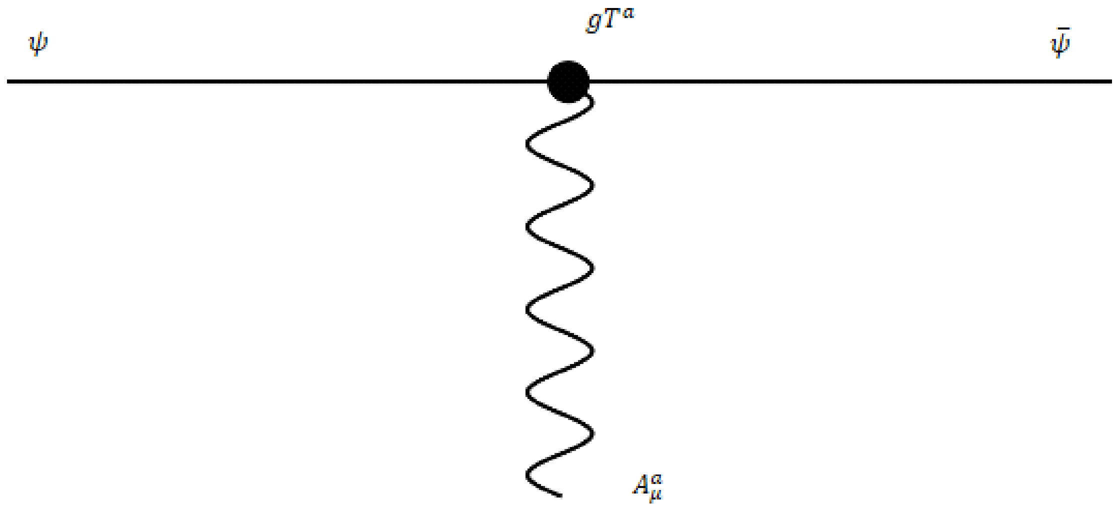
$$\boxed{J_\mu^a = \bar{\psi}\delta_\mu T^a\psi}$$

This is just like electrodynamics but now the current includes the group generator T^a

In electrodynamics, the photon field couples to the Fermions like $eA^\mu J_\mu = eA^\mu\bar{\psi}\delta_\mu\psi$

In non-abelian (group bigger than U(1)) there are several gauge boson fields A_μ^a, for each generator, with interactions

$$\boxed{gA^{\mu a}J_\mu^a = gA^{\mu a}\bar{\psi}\delta_\mu T^a\psi}$$



Group generator matrix T^a appears in the vertex

For weak SU(2), the A_μ^a are the 3 gauge bosons w^+, w^-, Z.

For colour SU(3), the A_μ^a are the 8 gluons

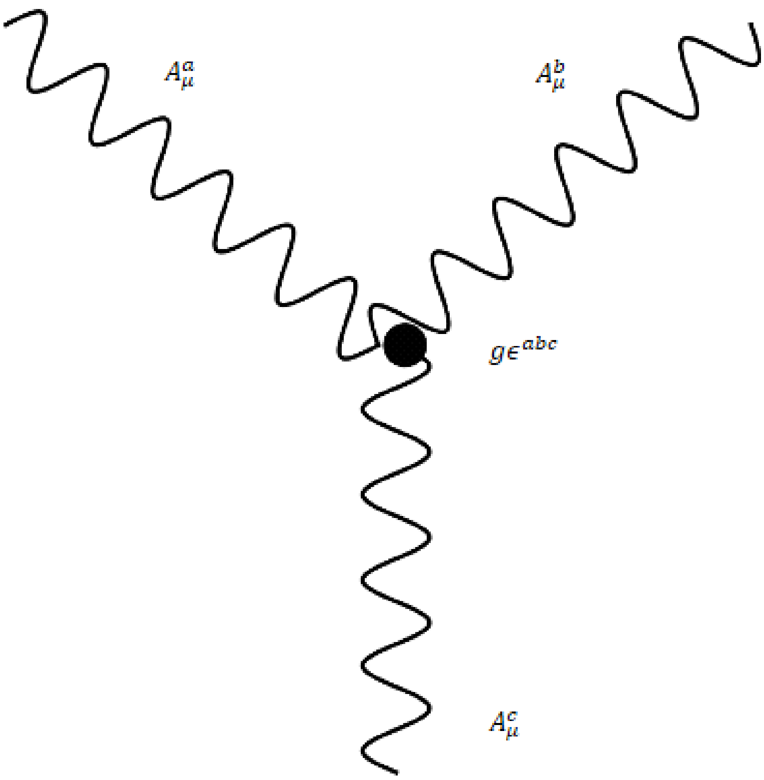
<u>Gauge-boson interactions</u>

U(1):- since the photon is neutral it does not couple to itself

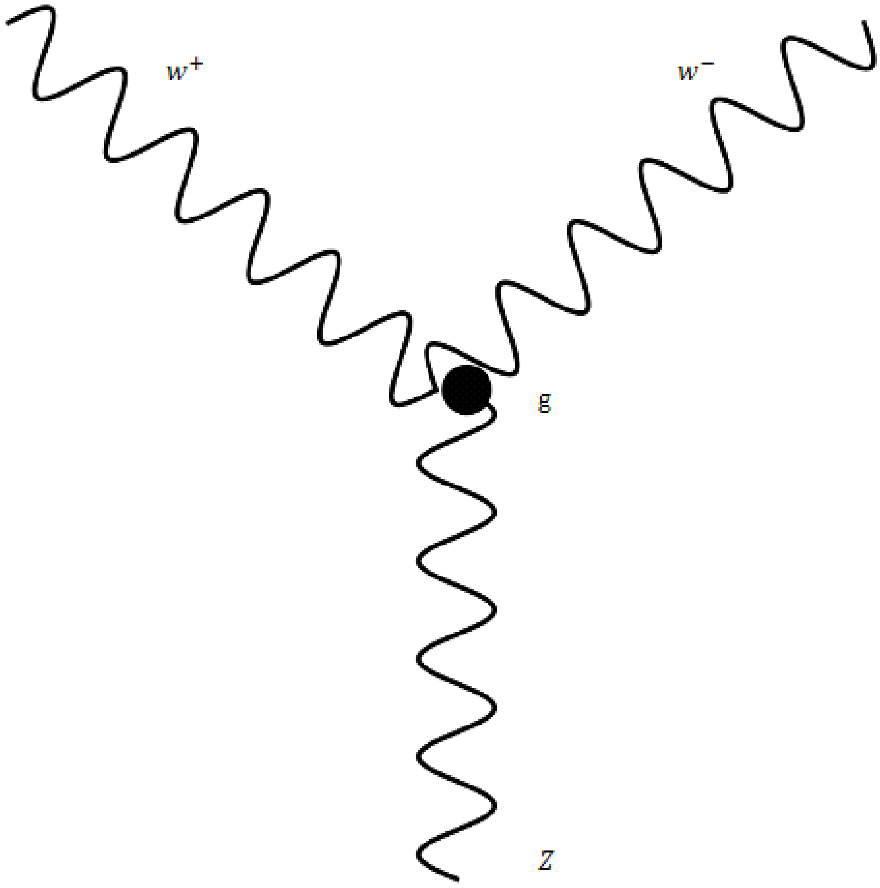
SU(2):- the gauge bosons do interact directly with themselves

Recall

$$[T^a, T^b] = i\epsilon^{abc}T^c$$



e.g. in SU(2)



3. Quarks, Leptons & Gauge Bosons

04 March 2013 10:39

All particles have "quantum numbers" related to spacetime or "internal" symmetries.

Spacetime → m (mass), s (spin)

Internal → Electric charge, "Weak charge", Colour, Lepton number, baryon number

Quarks & Leptons

We need to distinguish the helicity states L and R for quarks and leptons

Group into doublets and singlets according to SU(2):-

$\begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}, \quad \begin{pmatrix} \nu_{\mu L} \\ \mu_L^- \end{pmatrix}, \quad \begin{pmatrix} \nu_{\tau L} \\ \tau_L^- \end{pmatrix}$

$(\nu_{eR} \quad e_R^-), (\nu_{\mu R} \quad \mu_R^-), (\nu_{\tau R} \quad \tau_R^-)$

The R-handed neutrinos were not part of the minimal standard model which was developed when we believed netrinos were exactly massless.

Electron

$m = 0.55 MeV, \qquad s = \frac{1}{2}$

Lifetime $> 10^{26} yrs$

⇒stable

Muon

$m = 113.4 MeV, \qquad s = \frac{1}{2}$

Lifetime $2.2 \times 10^{-6} sec$

Decay $\mu \to e^- + \bar{\nu}_e + \nu_\mu$

(NB: Separate conservation of electron-type lepton no. and muon-type lepton no.)

Tau

$m = 1.77 GeV, \qquad s = \frac{1}{2}$

Lifetime $3 \times 10^{-13} sec$

Decays

$\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau, \qquad (17\%)$

$\tau^- \to e^- + \bar{\nu}_e + \nu_\tau, \qquad (17\%)$

Also

$\tau^- \to \pi^- + \nu_\tau, \qquad (10\%)$

$\tau^- \to \pi^- + \pi^0 + \nu_\tau, \qquad (25\%)$

-Branching Ratios

Quarks

Quarks match leptons

- "Quark-lepton universality"

The L-handed quarks form doublets

$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix}$

And R-handed singlets

$u_R \, d_R, \quad c_R \, s_R, \quad t_R \, b_R$

	Mass	Spin	Electric charge
Up	$m_u \approx 2MeV$	1/2	Q=2/3
Down	$m_u \approx 5MeV$	1/2	Q=-1/3

[NB Proton p=(uud), $m_p = 930MeV$ so the proton mass is almost entirely binding energy due to gluon interactions

Charm	$m_c \approx 1.25GeV$	1/2	Q=2/3
Strange	$m_s \approx 100MeV$	1/2	Q=-1/3

[NB Psi meson $\psi = \bar{c}c$ has $m_\psi \sim 3.1GeV$]

Top	$m_t \approx 175GeV$	1/2	Q=2/3
Bottom	$m_b \approx 4.2GeV$	1/2	Q=-1/3

[NB Upsilon meson $\Upsilon = \bar{b}b, m_\Upsilon \sim 10\text{GeV}$]

Note the huge hierarchy of quark masses

$$\frac{m_t}{m_u} \sim 10^5$$

Colour

Each type (flavour) of quark has a charge called "colour" that generalises electric charge

Electrodynamics U(1) Electric charge Q

Quantum Chromodynamics SU(3) Colour charge (3 colours R,G,B)

So for each q=u,d,c,s,t,b we have a colour triplet

$$\begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix}$$

But note colour charge has complicated addition rules. Colour charge is more like a vector

e.g. $p = u_R u_G d_B$ has zero colour

"Confinement" rule- only particles with zero colour exist as independent particles. Coloured particles are permanently bound ("Confined") inside baryons $(q_R q_G q_B)$ or mesons $(\bar{q}_{\bar{R}} q_R)$

Why 3 colours?

1) $\Omega^- = sss, \quad or\ \Delta^{++} = uuu$ violates pauli exclusion principle unless the quarks are different⇒ need (at least) 3 colours
2) $\pi^0 \to \gamma\gamma$ needs $N_c = 3$
3) Z width

Gauge Bosons

Photon

$$\gamma, \qquad m_\gamma = 0, \qquad s = 1, \qquad Q = 0$$

Photon interacts with electrically charged particles →QED, U(1)

Gluons

$$g, \qquad m_g = 0, \qquad s = 1, \qquad Q = 0$$

But gluons carry colour charge

$$8\ gluons \to QCD, SU(3)$$

Interactions in QCD is

$$\mathrm{L}_{\text{int}} = \Lambda^\mu \bar{\psi} \delta_\mu \mathrm{T}^a \psi$$

$$\bar{\psi} = \bar{q}_j$$

$$\psi = q_i$$

$$T^a = T^a_{ji}$$

Where T^a_{ji} = generator of SU(3) i.e. 3x3 hermitian matrix

For SU(3), there are 8 generators $(3^2 - 1 = 8)$

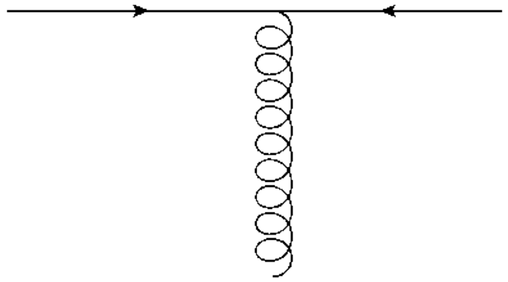
$$[T^a, T^b] = if^{abc}T^c$$

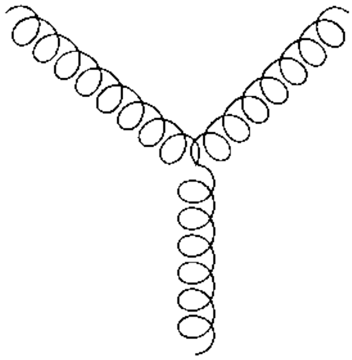
SU(3), structure constants

So there are 8 gauge bosons A^a_μ.

These are the gluons

⇒there are 2 types of interactions





3.2 Electroweak Interactions (Leptons)

Build electroweak theory on gauge interactions oof the general form

$gA_\mu^a \bar{\psi}\gamma^\mu T^a \psi$

g=coupling

A_μ^a =gauge boson

$\bar{\psi}\gamma^\mu T^a \psi$ =current

T^a=generator

Here we are only interested in interactions between particles so drop the spacetime indices

⇒Just write

$$\boxed{gA^a \bar{\psi} T^a \psi}$$

The unified theory of electrodynamics and weak interactions is described by a gauge theory with group $SU(2)_L \times U(1)_Y$

There are 3 gauge bosons for $SU(2)_L$ called W^a, $a = 1,2,3$ corresponding to generators T^a

These only interact with L-handed fermions. So this interaction distinguishes L and R helicity, so violates Parity. (Weak interaction parity violation discovered in ~1956)

The other gauge group $U(1)_Y$ has a gauge boson B, coupling to the quantum number Y (weak hypercharge) of the fermions.

The electroweak theory (Weinberg, Salam 1967) has interaction Lagrangian :-

$$L_{\text{int}} = g(\bar{\nu}_{eL} \quad e_L^+)T^a \begin{pmatrix}\nu_{eL}\\ e_L^-\end{pmatrix} W^a + g'\bar{\nu}_{eL} Y_\nu \nu_{eL} + g' e_L^+ Y_{e_L} e_L^- + g' e_R^+ Y_{e_R} e_R^-$$

+ same for μ and τ generations

Where

$g = SU(2)_L$ coupling

$g = U(1)_Y$ coupling

And

$Y_\nu = -1$ for ν_{eL}

$Y_{e_L} = -1$ for e_l

(same)

$Y_{e_R} = -2$ for e_R

NB:

We have not included a ν_{eR}.

This is the original standard model with massless neutrinos.

Look at w^a interactions first.

$$L_{\text{int}} \sim \frac{g}{2}(\bar{\nu}_{eL} \quad e_L^+) w^a \sigma^a \begin{pmatrix}\nu_{eL}\\ e_L^-\end{pmatrix}$$

Since $T^a = \frac{1}{2}\sigma^a$

Where

$$\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$L = \frac{g}{2}(\bar{\nu}_{eL} \quad e_L^+)\begin{pmatrix} w^3 & w^1 - iw^2\\ w^1 + iw^2 & -w^2\end{pmatrix}\begin{pmatrix}\nu_{eL}\\ e_L^-\end{pmatrix}$$

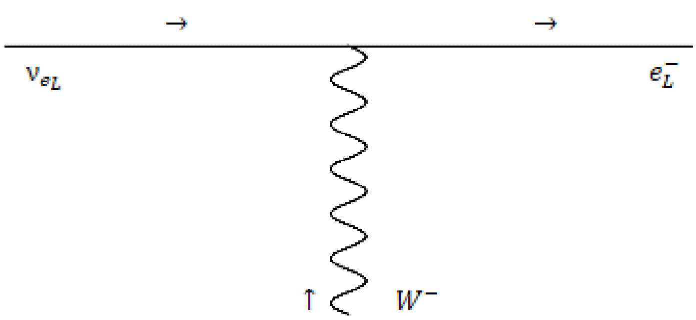
$$= \frac{g}{\sqrt{2}}(W^+ \bar{\nu}_{eL} e_L^- + W^- e_L^+ \nu_{eL})$$

((2) + (1))

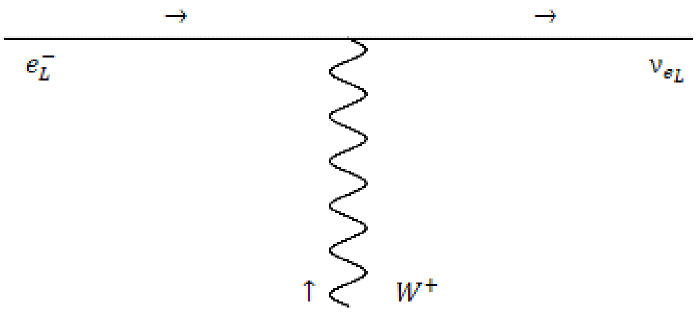
Defining

$$W^\pm = \frac{1}{\sqrt{2}}(W^1 \mp iW^2)$$

These interactions give vertices on Feynman diagrams:-



(1)



(2)

Notes:

Denotes flow of lepton number. Must conserve lepton number and electric charge

Coupling strength

$$= \frac{g}{\sqrt{2}}$$

$$L_{\text{int}} = \frac{g}{\sqrt{2}}(W^+\bar{\nu}_{eL}e_L^- + W^- e_L^+ \nu_{eL}) + \frac{g}{2}W^3(\bar{\nu}_{eL}\nu_{eL} - e_L^+ e_L^-)$$

Rule

1) View the 3 fields in L_{int} asincoming into interaction vertex
2) Incoming particle is equivalent to outgoing antiparticle
3) Check electric charge + Lepton number conservation at vertices

Next, consider couplings of B field.

$$L_{\text{int}} \sim \frac{1}{2}g' B\left[Y_{\nu_{eeL}}\bar{\nu}_{eL}\nu_{eL} - Y_{e_L}e_L^+ e_L^- + Y_{e_R}e_R^+ e_R^-\right]$$

$-1, \qquad -1, \qquad -2$

So the "neutral" interactions are

$$L_{\text{int}}^{\text{neutral}} \sim \bar{\nu}_{eL}\nu_{eL}\left(\frac{g}{2}W^3 - \frac{g'}{2}B\right) + e_L^+ e_L^-\left(-\frac{g}{2}W^3 - \frac{g'}{2}B\right) + e_R^+ e_R^-(-g'B)$$

Weinbergz-Salam-Glashow, the physical photon (A) and Z are linear combinations of B and W^3.

Define

$$B = A\cos\theta_w - Z\sin\theta_w$$
$$W_3 = A\sin\theta_w - Z\cos\theta_w$$

NB

$\begin{pmatrix} B \\ W^3 \end{pmatrix}$ related to $\begin{pmatrix} A \\ Z \end{pmatrix}$ by orthogonall matrix

The mixing angle θ_w is called the weinberg angle (or "Weak angle")

Let $g' = g\tan\theta_w$

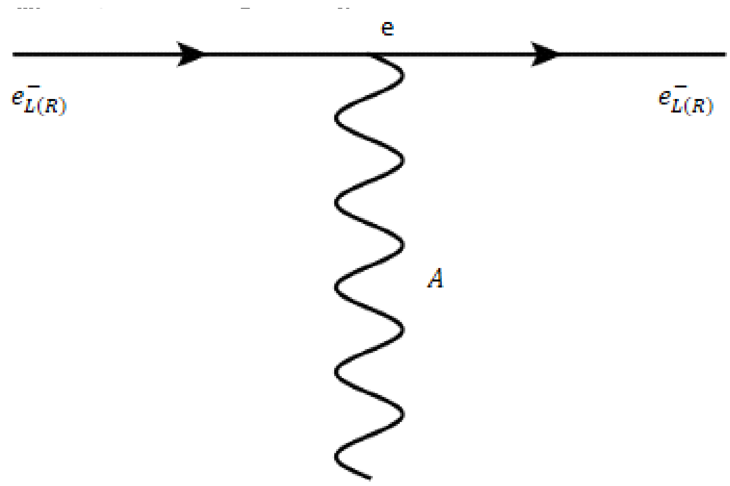
Substituting for B, W_3 in terms of A,Z we get

$$L_{\text{int}}^{\text{neutral}} \sim \frac{1}{2}gA[\bar{\nu}_{eL}\nu_{eL}(\sin\theta_w - \tan\theta_w\cos\theta_w) + e_L^+ e_L^-(-\sin\theta_w - \tan\theta_w\cos\theta_w) + e_R^+ e_R^-(-2\tan\theta_w\cos\theta_w)]$$

$$+ \frac{1}{2}gZ[\bar{\nu}_{eL}\nu_{eL}(\cos\theta_w - \tan\theta_w\sin\theta_w) + e_L^+ e_L^-(-\cos\theta_w - \tan\theta_w\sin\theta_w) + e_R^+ e_R^-(-2\tan\theta_w\sin\theta_w)]$$

The mixing angle $\tan\theta_w = g'/g$ was chosen so that A has the correct couplings to be the photon:-

$$L_{\text{int}}^{A} \sim -g\sin\theta_w A(e_L^+ e_L^- - e_R^+ e_R^-)$$



That is, photon couples to charged electron (same way for e_L^- and e_R^- because electrodynamics conserves parity) nad not to the neutral neutrino. Coupling strength is identified as $e = g\sin\theta_w$

Parameters:-

Started with g (For $SU(2)_L$) and g' (for $U(1)_Y$)

Swapped for two new parameters

e (electric coupling)

$\sin\theta_w$ (weinberg angle)

Remaining interactions with the Z are

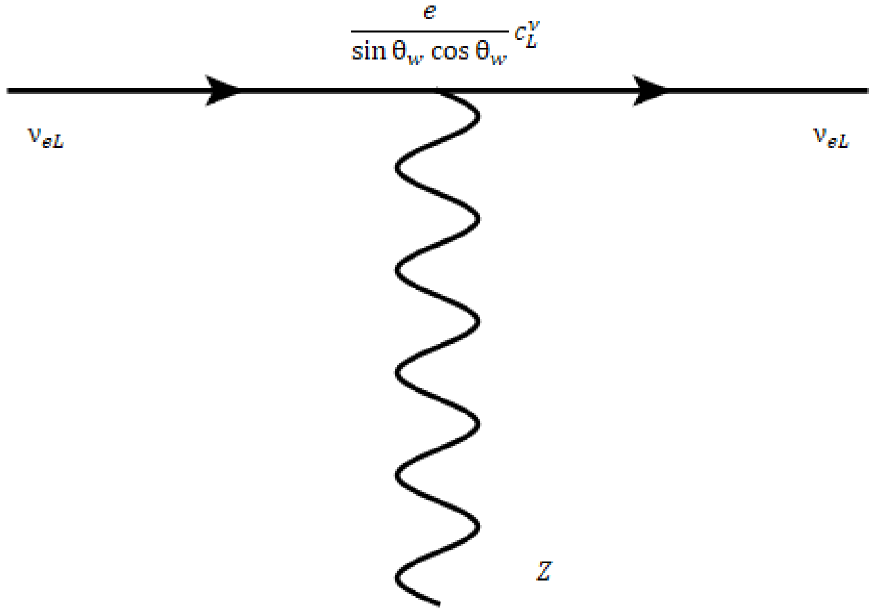
$$L^Z_{\text{int}} \sim \frac{e}{\sin\theta_w \cos\theta_w} Z(c_L^\nu \bar{\nu}_{eL}\nu_{eL} + c_L^e e_L^+ e_L^- + c_R^e e_R^+ e_R^-)$$

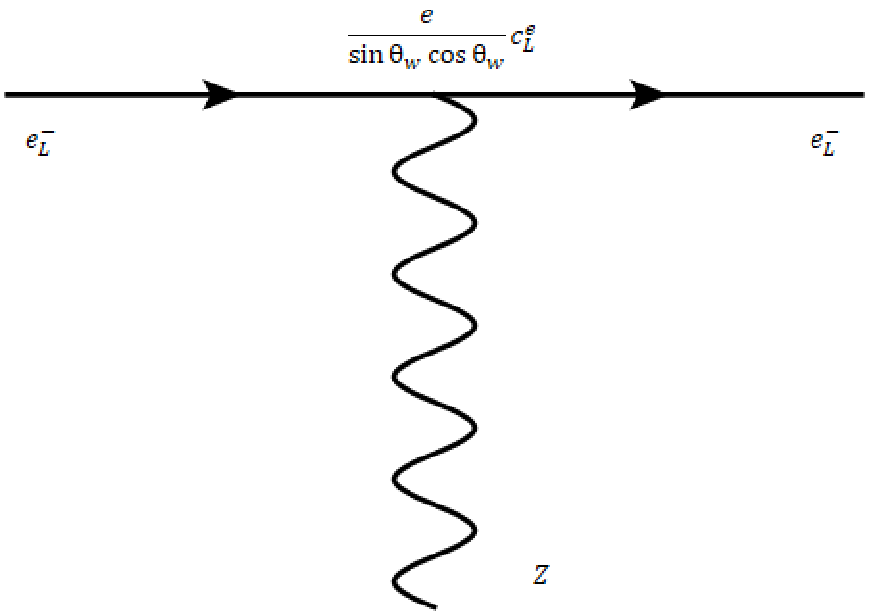
Where

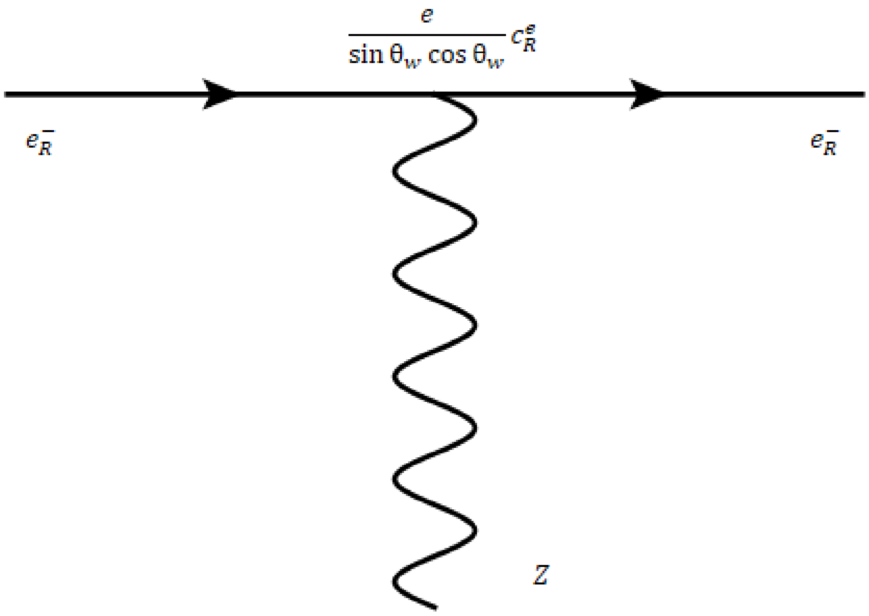
$$c_L^\nu = \frac{1}{2}$$

$$c_L^e = -\frac{1}{2} + \sin^2\theta_w$$

$$c_R^e = \sin^2\theta_w$$







Notice that the Z couples to the neutrino as well as electrons and coupling to e_L^- and e_R^- are <u>different</u> $\Leftrightarrow$ parity violation

3.3: Electroweak Interactions at Low Energy

The $SU(2)_L \times U(1)_y$ action gives the interactions in Feynman diagrams. This shows which reactions can take place

<u>Examples</u>

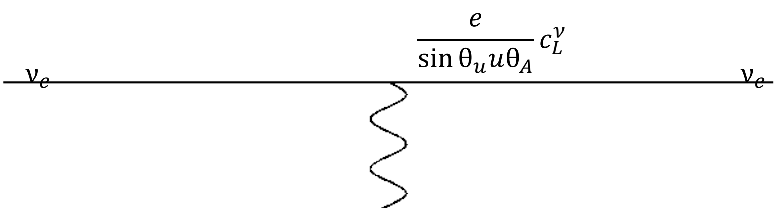
1) $\nu_\mu + e^- \to \nu_e + \mu^-$

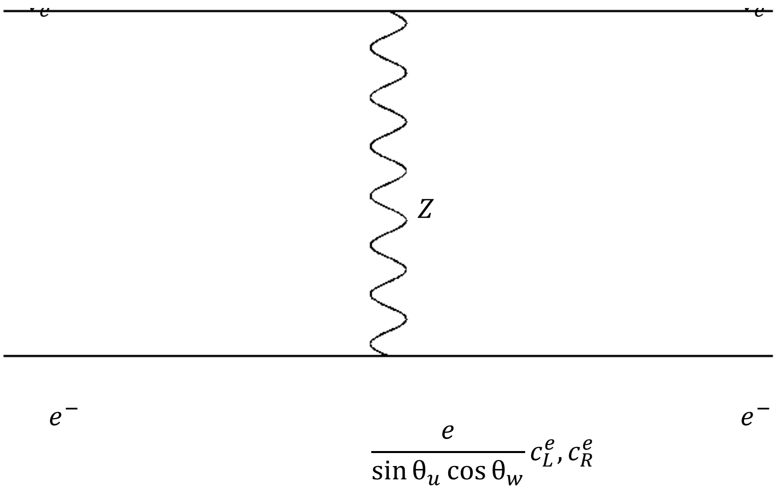
   Lepton numbers

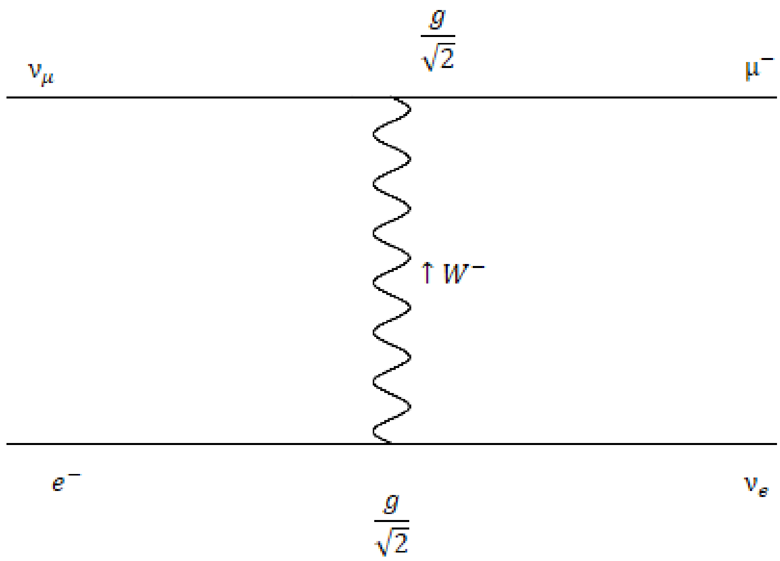
   $L_\mu = 1, L_e = 1 \Rightarrow L_\mu = 1, L_e = 1$

   Charge

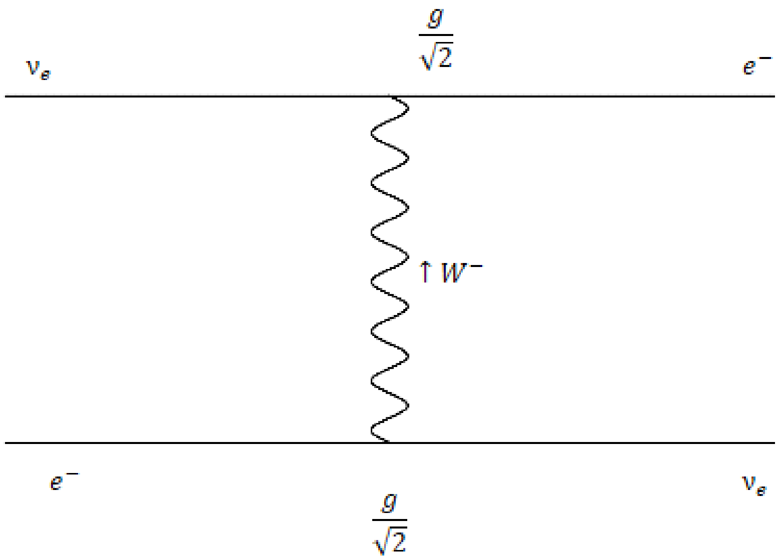
   $Q = 0, \qquad -1 \Rightarrow 0, \qquad -1$



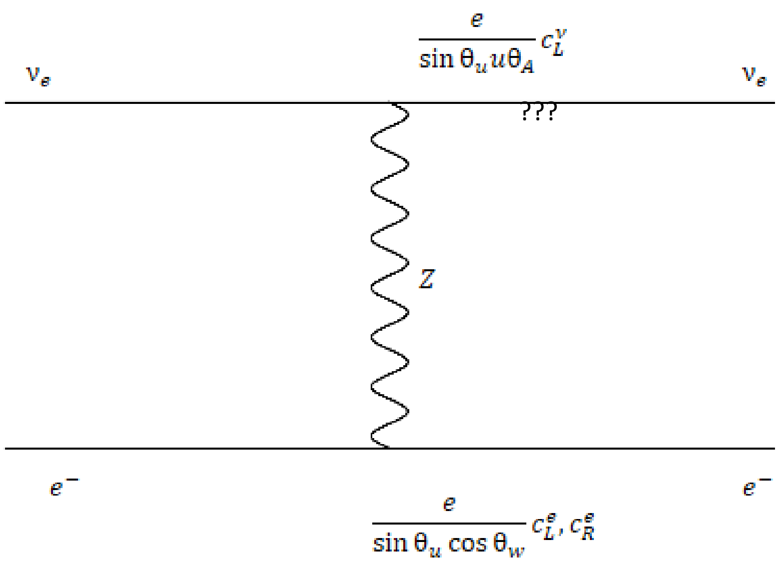




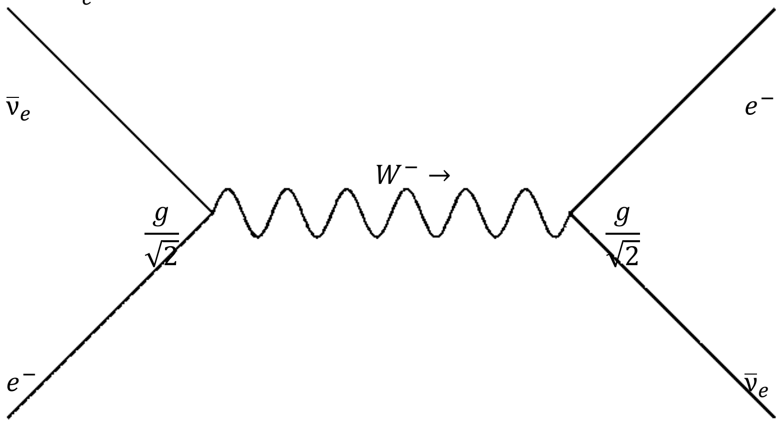
2) $\nu_e + e^- \to \nu_e + e^-$

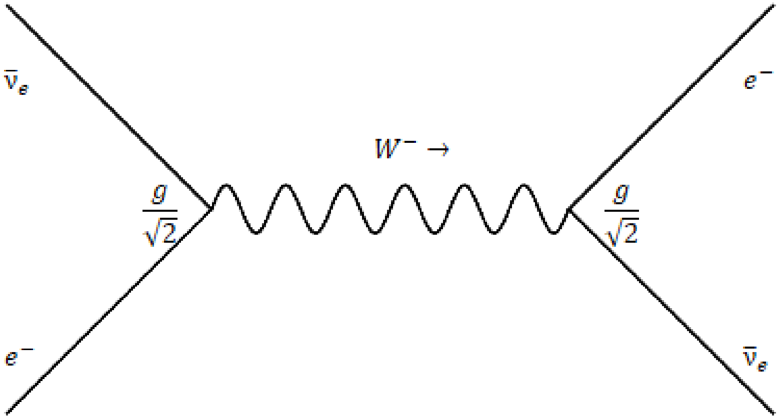


Another possible diagram involves Z exchange, i.e. "neutral current" reaction

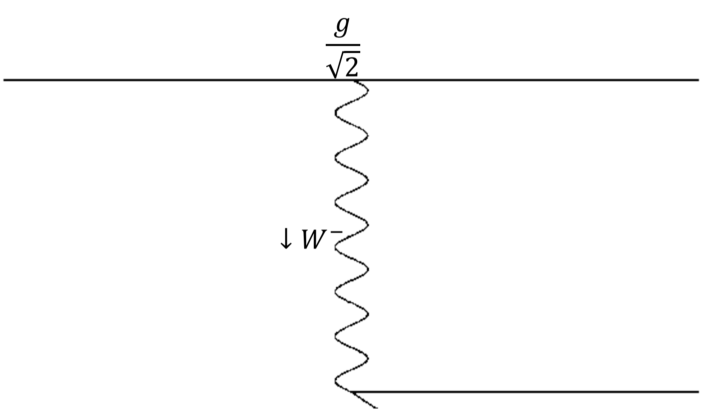


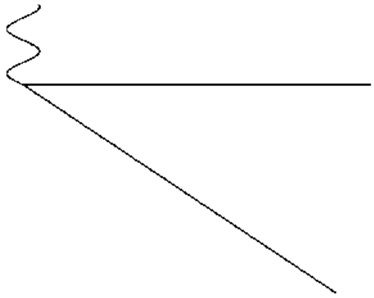
3) $\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$





4) $\mu^- \to e^- + \nu_e + \nu_\mu$





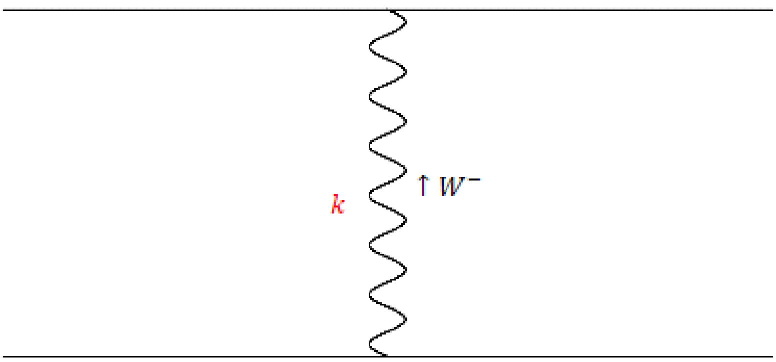
~~~

5) $e^+ + e^- \to e^+ + e^-$

~~~

Include propagators:-

$\nu_\mu + e^- \to \nu_e + \mu^-$



W propagator

$$\sim \frac{1}{k^2 + m_w^2}$$

Where $k + p_1 = p_3, \quad p_2 = k + p_4$

Cons,

$$p_1 + p_2 = p_3 + p_4$$

Diagram contributes a factor

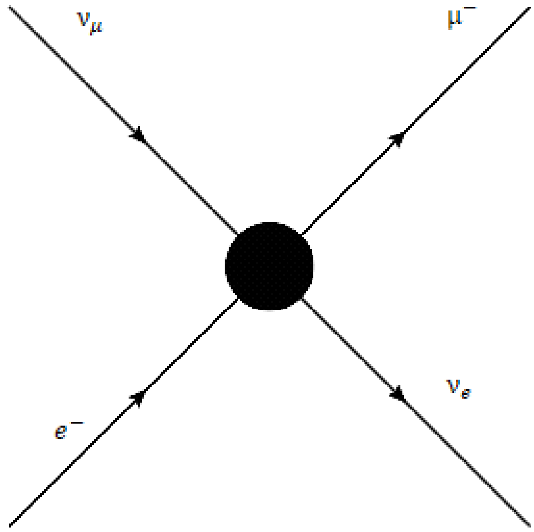
$$\left(\frac{g}{\sqrt{2}}\right)^2 \frac{1}{k^2 + m_w^2}$$

To a calculation of cross-section for $\nu_\mu e^- \to \nu_e \mu^-$

Now for energies <<m_w, m_z. The momentum-transfer dependence in the propagator is small compared to m_w^2, m_z^2. So at low energies, diagram contribution is approx

$$\left(\frac{g}{\sqrt{2}}\right)^2 \frac{1}{m_w^2}$$

This means the reaction looks like a point interaction amongst the 4 fermions:-



G_F =Fermi coupling
(known. Measured in β-decay)

Strength of effective 4-Fermi vertex was called

$$\frac{4G_F}{\sqrt{2}}$$

So

$$\frac{4G_F}{\sqrt{2}} = \frac{g2}{8m_w^2} = \frac{e^2}{8m_w^2 \sin^2\theta_w}$$

$\frac{4G_F}{\sqrt{2}}$ =measured in low-energy weak interactions

e =electric coupling

Measure $\sin^2\theta_w$ by comparing different low-energy weak interactions,

So this becomes a prediction for the w mass based only on low-energy data!

$$\Rightarrow \boxed{m_w^2 = \frac{e^2}{8\sin^2\theta_w}\frac{\sqrt{2}}{G_F}}$$

Numbers:-

$\sin^2\theta_w = 0.226 \pm 0.005$

$G_F = 1.166 \times 10^{-5} GeV^{-2}$

$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$

$\Rightarrow m_w = 78.4 \pm 0.9 GeV$

Similarly, using relation

$m_w = m_z \cos\theta_w$

$\Rightarrow m_z = 89.2 \pm 0.8 GeV$

Since this is a prediction, we knew the energy required for accelerators to discover W and Z

First $Sp\bar{p}S$ at CERN ~ 1983

Then LEP at CERN

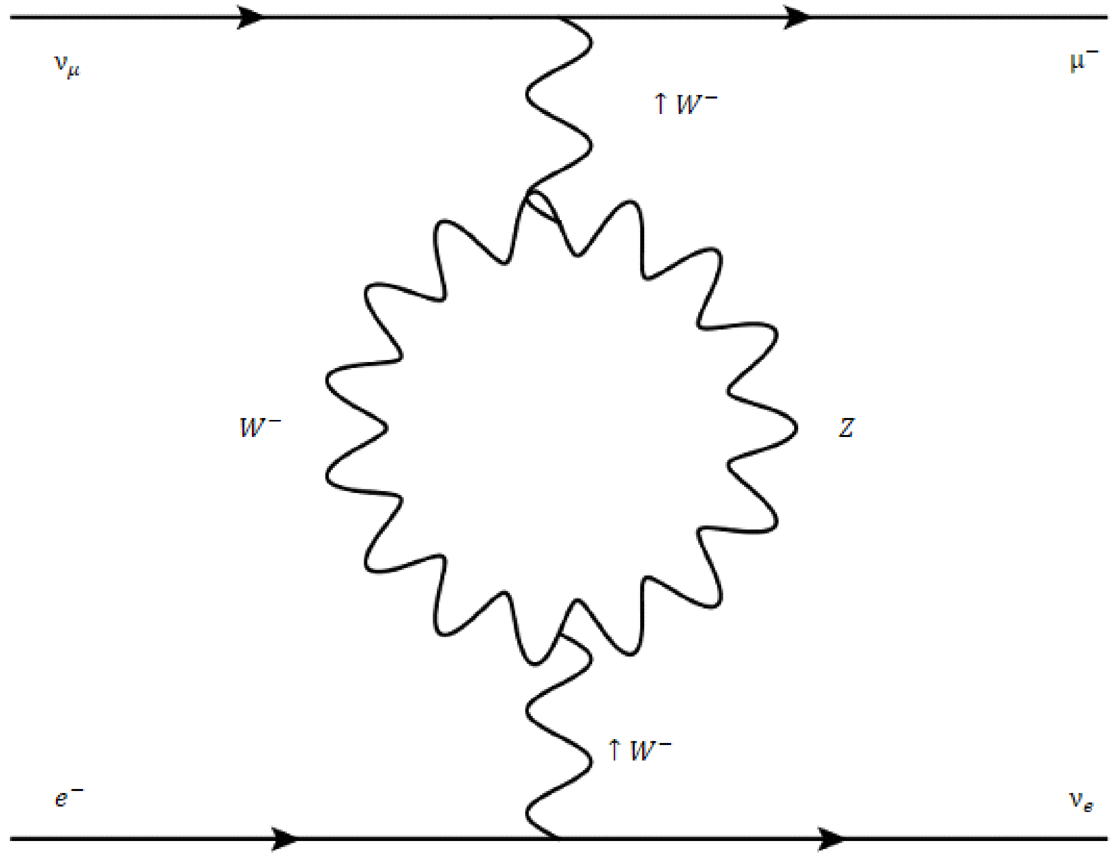
e^+e^- accelerator with $E_{beam} \sim \frac{1}{2} m_z \sim 45 GeV$

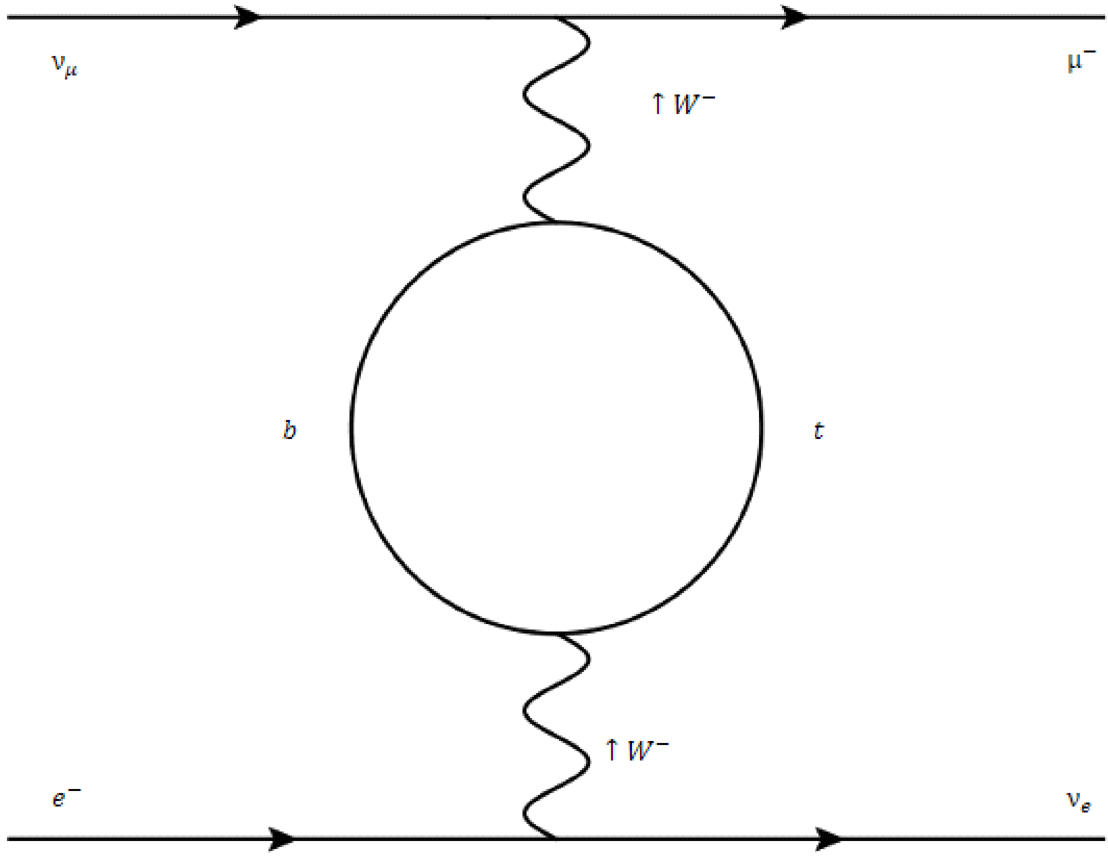
In fact the precision measurements of m_w and m_z at LEP and Tevatron are slightly different

$m_w = 80.403 \pm 0.029$ GeV

$m_z = 91.188 \pm 0.002$ GeV

Discrepancy is due to omitting more complicated Feynman diagrams





+ others

Note that even at energies $\ll m_t$, the top quark still contributes as a propagator in an internal loop.
So precision measurements at low energies can predict m_t.
Prediction for precision measurements at LEP.

$\Rightarrow m_t = 172.3 \pm 10.2$ GeV

Direct observation at Tevatron (1995)

$\Rightarrow m_t = 174.2 \pm 3.3$ GeV

The same method was used to constrain the mass of the Higgs before its direct observation.
Can also be used to search for new particles, e.g. supersymmetric particles.

3.4 Electroweak Interactions For Quarks

This is very similar to leptons:-

3 L-handed generations:

$$\begin{pmatrix} u_L \\ d'_L \end{pmatrix}, \quad \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \quad \begin{pmatrix} t_L \\ b'_L \end{pmatrix}$$

Plus R-handed

$$u_R, d_R, \quad c_R, s_R, \quad t_R, b_R$$

NB: d', s', b' are linear combinations of physical quarks d,s,b

Discuss quark mixing later

The $SU(2)_L \times U(1)_Y$ interaction Lagrangian is (for 1st generation- others identical)

$$L_{int} = gW^a(\bar{u}_L \quad \bar{d}_L)T^a \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

$$+\frac{1}{2}g'B\,\bar{u}_L Y_{u_L} u_L$$
$$+\frac{1}{2}g'B\,\bar{d}_L Y_{d_L} d_L$$
$$+\frac{1}{2}g'B\,\bar{u}_R Y_{u_R} u_R$$
$$+\frac{1}{2}g'B\,\bar{d}_R Y_{d_R} d_R$$

Quantum nos chosen to be $Y_{u_L} = Y_{d_L} = \frac{1}{3}, \;\; Y_{u_R} = \frac{4}{3}, \;\; Y_{d_R} = -\frac{2}{3}$

Now calculate the interactions exactly as for the leptons (problem sheet 2)

Results

Charged weak:-

$$L_{int}^{w^\pm} = \frac{g}{\sqrt{2}}\left(W^+\bar{u}_L d_L + W^-\bar{d}_L u_L\right)$$

Electromagnetism:-

$$L_{int}^{A} = eA\left(\frac{2}{3}\bar{u}_L u_L + \frac{2}{3}\bar{u}_R u_R - \frac{1}{3}\bar{d}_L d_L - \frac{1}{3}\bar{d}_R d_R\right)$$

The u_L 2/3→ $T_3 + \frac{1}{2}Y = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

Neutral Weak

$$L_{int}^{Z} = \frac{e}{\sin\theta_u \cos\theta_w} Z\left(c_L^u \bar{u}_L u_L + c_L^d \bar{d}_L d_L + c_R^u \bar{u}_R u_R + c_R^d \bar{d}_R d_R\right)$$

Where

$$c_L^u = \frac{1}{2} - \frac{2}{3}\sin^2\theta_w$$
$$c_L^d = -\frac{1}{2} + \frac{1}{3}\sin^2\theta_w$$
$$c_R^u = -\frac{2}{3}\sin^2\theta_w$$
$$c_R^d = \frac{1}{3}\sin^2\theta_w$$

Note that the Y quantum numbers are chosen so that the photon A couples to quarks with coupling = electric charge Q

Find $Q = T_3 + \frac{1}{2}Y$

Examples of Feynman diagrams

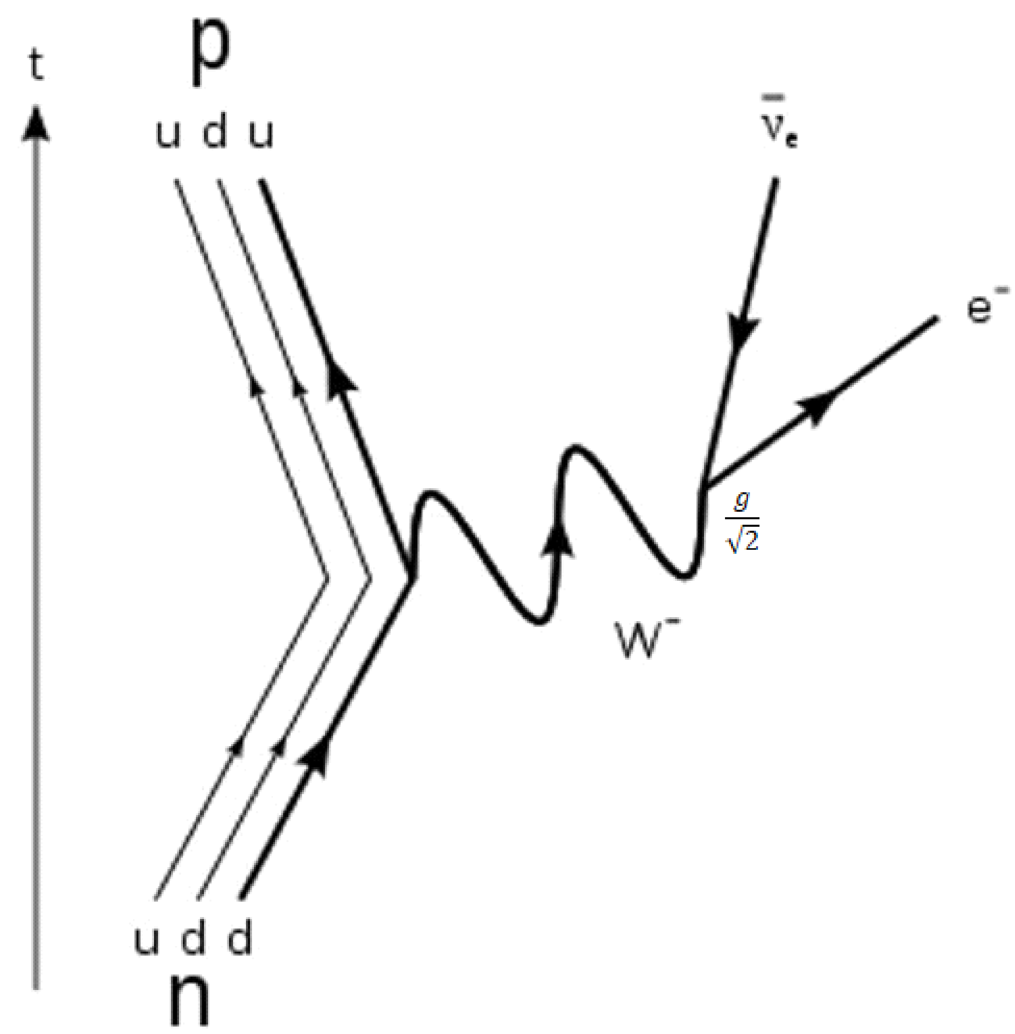
Low energies

$$n \to p + e^- + \bar{\nu}_e$$

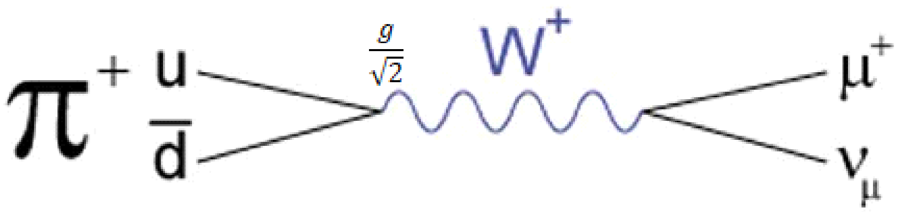
β −decay

i.e.

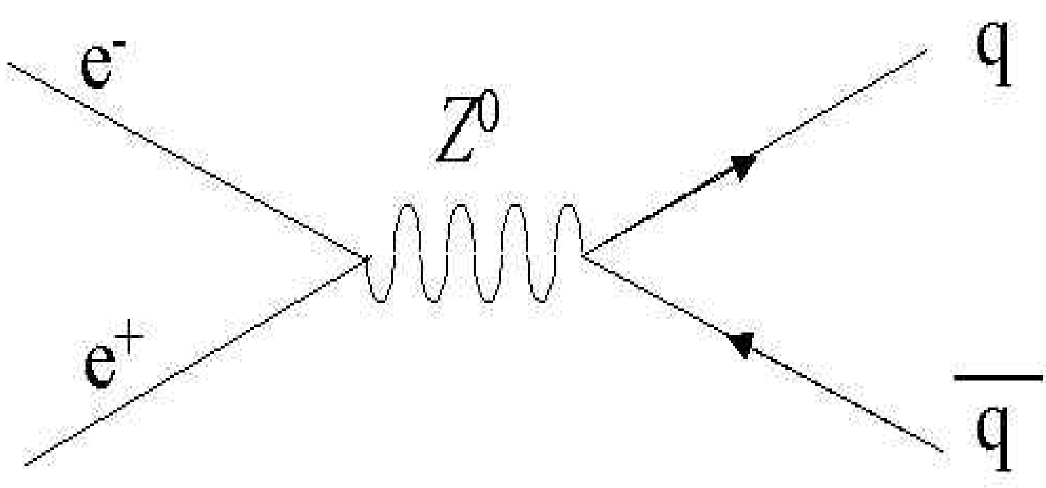
$$d \to u + e^- + \bar{\nu}_e$$



$\pi^{\pm} \to \mu^{+} + \nu_{\mu}$



High energies (LEP)
$e^{+}e^{-} \to Z \to \bar{q}q$ (or lepton/antilepton pair)



Quark mixing and CKM matrix

Recall $\pi^{\pm} > \mu^{+} + \nu_{\mu}$

~feyn diag, $\frac{g}{\sqrt{2}} V_{ud}$ at vertex~

However $K^{\pm} > \mu^{+} + \nu_{\mu}$ *is alos observed*

~feyn diag, $\frac{g}{\sqrt{2}} V_{us}$~

But this vertex looks as if it doesn't exist in electroweak lagrangian

Resolution :- The d', s', b' quarks in the electroweak lagrangian are not physical quarks d,s,b

These quarks mix:-

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}$$

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = V$$

Cabibbo-Kobayashi-Maskowo (CKM) matrix

No proof

CKM matrix V is unitary.

Actually has 4 parameters- 3 real mixing angles + 1 imaginary parameter

$\rightarrow$CP violation

Simpler to just consider the first two generations

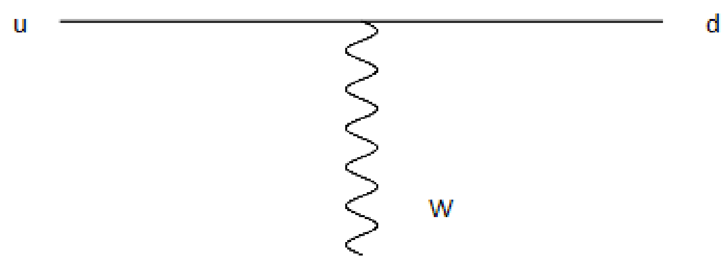
Then V reduces to a 2x2 matrix, with 1 real mixing angle, $\theta_c = \text{Cabibbo Angle}$

$$\begin{pmatrix} d \\ s \end{pmatrix} = \begin{pmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d' \\ s' \end{pmatrix}$$

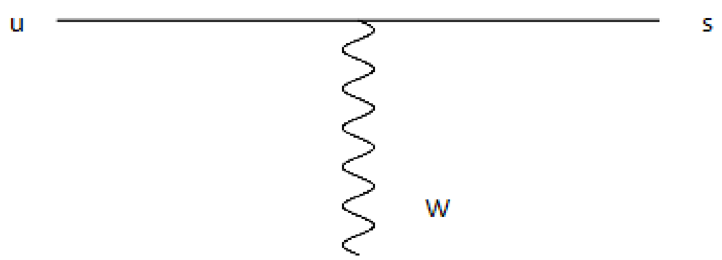
$\theta_c \approx 12.7°$

$\sin\theta_c \approx 022$

So the vertices are



$$\frac{g}{\sqrt{2}} \cos\ \theta_c$$



$$\frac{g}{\sqrt{2}} \sin\theta_c$$

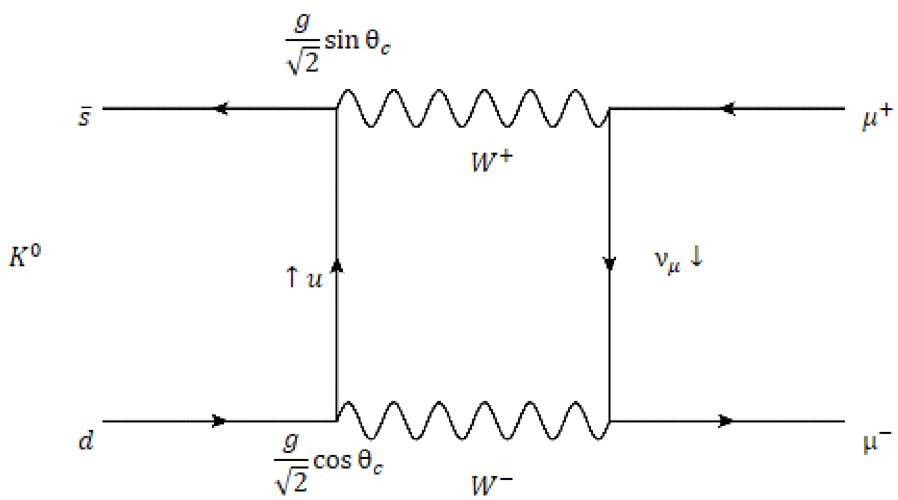
Decay Ratio

$$\frac{\Gamma[K^{+} \rightarrow \mu^{+}\nu_{\mu}]}{\Gamma[\pi^{+} \rightarrow \mu^{+}\nu_{\mu}]} = \frac{\sin^2\theta_c}{\cos^2\theta_c} = \tan^2\theta_c$$

Up to dynamical factors

<u>"Flavour-changing neutral currents"</u>

FCNCs are strongly suppressed
Eg
$$\frac{\Gamma[K^0 \to \mu^+\mu^-]}{\Gamma[K^0 \to \text{Anything}]} \sim 10^{-8}$$
$K^0 \to \mu^+\mu^-$ has $\Delta s = 1$, where s=strangeness



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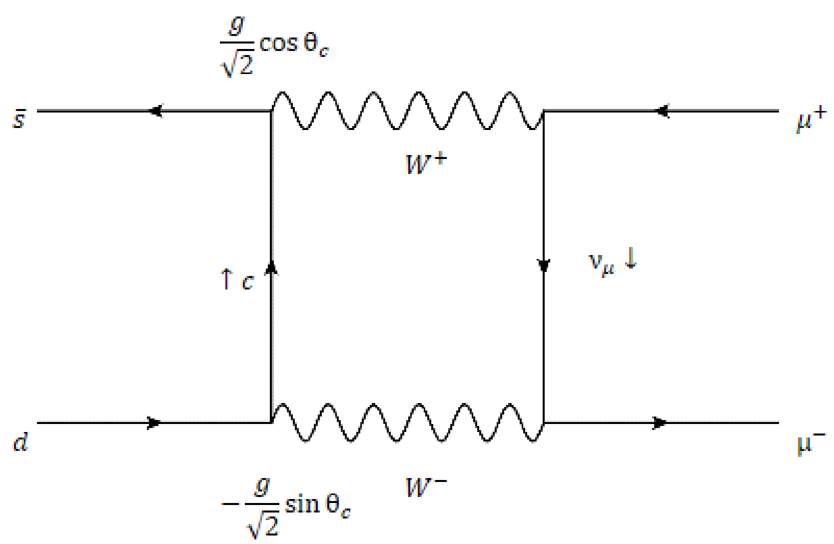

$$\sim \frac{g^4}{2}(\cos\theta_c \sin\theta_c - \cos\theta_c \sin\theta_c) = 0$$
(because c only couples to s' in Lagrangian)

Now - Feynman diagrams are amplitudes
$= Probability = |\sum ampl|^2$
So the two feynman diagrams cancel, i.e. amplitudes interfere (GIM mechanism)
$\Rightarrow$Total probability for $K^0 \to \mu^+\mu^-$ is zero $\boxed{\text{by Glashow-Lliopoulos-Maini}}$
This was important historically (before 1974) as motivation for the prediction of the charm quark

~~~
Klein-Gordon field:-
~~~

$$S_0 = \int d^4x(-\delta^\mu \phi^* \delta_\mu \phi - m^2 \phi^* \phi)$$

Again, let $\delta_\mu \to D_\mu = \delta_\mu - iA_\mu$

~~

First, we need a bit more theory about scalar fields. This involves "covariant derivatives"
Dirac:-

$$S_0 = \int d^4x\, \overline{\psi}(i\gamma^\mu \delta_\mu + m)\psi$$

Let

$$\delta_\mu \to D_\mu = \delta_\mu - iA_\mu$$

$$\Rightarrow S = \int d^4x\, \overline{\psi}(i\gamma^\mu D_\mu + m)\psi = S_0 + \int d^4x\, A_\mu \overline{\psi}\gamma^\mu \psi$$

D_μ =covarient derivative

$\overline{\psi}\gamma^\mu \psi = J^\mu$

So for the scalar field, gauge invariance implies an extra 4-point interaction

$A_\mu A^\mu \phi^* \phi$

As well as the gauge field-current interaction $A_\mu J^\mu$

~~

Higgs mechanism for $SU(2)_L \times U(1)$ theory

Higgs field in electroweak theory is a complex, $SU(2)_L$ doublet, scalar field
Assign $Y = 1$
Remember

$$Q = T_3 + \frac{1}{2}Y$$

$$\to \phi = \binom{\dots}{\dots} \begin{array}{l} \to \quad T_3 = \frac{1}{2}, \quad Y = 1 \Rightarrow Q = 1 \quad \textit{charged} \\ \to \quad T_3 = -\frac{1}{2}, \quad Y = 1 \Rightarrow Q = 0 \quad \textit{Electric charge neutral} \end{array}$$

Since ϕ is complex, there are 4 real fields
Of these, 2 are charged, 2 are neutral
For the Higgs mechanism, the field that has a non-zero vacuum expectation value must be neutral (to keep charge conservation)
So we write

$$\phi = \begin{pmatrix} \phi^1 + i\phi^2 \\ v + H + \phi^3 \end{pmatrix}$$

Where

$$\phi^\pm = \frac{1}{\sqrt{2}}(\phi^1 \pm i\phi^2), \phi^3 \; neutral$$

And v= vacuum expectation value of Higgs field
H is the Higgs boson field

Goldstone bosons

The fields ϕ^+, ϕ^-, ϕ^3 are called goldstone bosons. They would correspond to massless spin 0 particles
But, in the electroweak theory with Higgs mechanism, they combine with the gauge fields W^+, W^-, Z to produce massive gauge bosons

This works because a massive spin 1 particle has 3 helicity states, whereas a massless spin 1 particle has only 2, the extra state is provided by the Goldstone boson.
So in the final spectrum, the 3 Goldstone bosons do not appear - they are just the extra helicity states of the massive gauge bosons W^+, W^-, Z

$SU(2) \times U(1)_L$ Higgs mechanism

$$\phi = \begin{pmatrix} \phi^1 + i\phi^2 \\ v + H + \phi^3 \end{pmatrix}$$

Goldstone bosons disappear and give longitudinal polarisation states of the massive W and Z
Neglect goldstone bosons from now on
Take

$$\phi = \begin{pmatrix} 0 \\ v+H \end{pmatrix}$$

Covariant derivation

$$D_\mu = \delta_\mu - iA_\mu \to_{SU(2)_L \times U(1)} \left(\delta_\mu + igT^a W_\mu^a + i\frac{g'}{2} YB_\mu\right)$$

$$A_\mu A^\mu \phi^* \phi \to \phi^\dagger \left(gT^a W_\mu^a + \frac{g'}{2} YB_\mu\right)\left(gT^a W^{\mu a} + \frac{g'}{2} YB^\mu\right)\phi$$

To get the Higgs (H field) interactions we need to evaluate

$$\phi^\dagger \left(gT^a W_\mu^a + \frac{g'}{2} YB_\mu\right)\left(gT^a W^{\mu a} + \frac{g'}{2} YB^\mu\right)\phi$$

Evaluate

$$gT^a W^a + \frac{1}{2} g' B = \frac{1}{2}\begin{pmatrix} gW^3 + g'B & g(W^1 - iW^2) \\ g(W^1 - iW^2) & -gW^3 + g'B \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2g\sin\theta_W A + \frac{g}{\cos\theta_w}(1 - 2\sin^2\theta_w)Z & \sqrt{2} g W^+ \\ \sqrt{2} g W^- & \frac{g}{\cos\theta_w} Z \end{pmatrix}$$

$$\left(gT^a W^a + \frac{1}{2} g' B\right)^2 = \frac{1}{4}\begin{pmatrix} X & X \\ X & 2g^2 W^+ W^- + \frac{g^2}{\cos^2\theta_w} Z^2 \end{pmatrix}$$

Next

$$\phi^\dagger (\square)^2 \phi$$

$$= \frac{1}{2}(0 \quad v+H)\begin{pmatrix} X & X \\ X & \boxed{\square} \end{pmatrix}\begin{pmatrix} 0 \\ v+H \end{pmatrix}$$

$$= \frac{1}{2}(v+H)^2\left(\frac{1}{2} g^2 W^+ W^- + \frac{g^2}{4\cos^2\theta_w} Z^2\right)$$

Summarising, the coupling of $W^\pm, Z$ to the Higgs field H and expectation value v is just

$$L_{int} = \frac{1}{2}(v+H)^2\left(\frac{1}{2} g^2 W^+ W^- + \frac{g^2}{4\cos^2\theta_w} Z^2\right)$$

NB: H does not couple to photon field A_μ because it has charge Q=0
Two point interactions,

$$L_{int}^{(2)} = \frac{1}{2}\left(\frac{1}{2} g^2 v^2 W^+ W^- + \frac{g^2 v^2}{4\cos^2\theta_w} Z^2\right)$$

Write as

$$L_{int}^{(2)} = m_w^2 W^+ W^- + \frac{1}{2} m_z^2 Z^2$$

Identify

$$m_Z = \frac{g}{2\cos\theta_w} v$$

$$m_w = \frac{g}{2} v$$

We see that the interaction with the Higgs field vacuum expectation value gives the $W^\pm, Z$ bosons a mass!
Note relation

$$\frac{m_W}{m_Z \cos\theta_w} = 1$$

This is characteristic of the Higgs $SU(2)_L \times U(1)$ model
Recall

$$m_W = 80.4 GeV$$

$$m_Z = 91.2 GeV$$

$$\sin^2\theta_w = 0.226$$

Note how 2-pt interactions give mass term in a propagator:-
If $L_{int} \sim \frac{1}{2} m^2 A^2$
~~~
~~~

$$= -\frac{1}{p^2} + \frac{-1}{p^2}m^2\frac{-1}{p^2} + \frac{-1}{p^2}m^2\frac{-1}{p^2}m^2\frac{-1}{p^2}$$
$$= \frac{-1}{p^2}\left(1 + \frac{m^2}{p^2}\right)^{-1}$$
$$= -\frac{1}{p^2 + m^2}$$
Which is the propagator for a massive particle

Two-point interactions,
$$L_{int}^{(2)} = \frac{1}{2}\left(\frac{1}{2}g^2v^2W^+W^- + \frac{g^2v^2}{4\cos^2\theta_w}Z^2\right)$$
Three-point interactions:-
$$L_{int}^{(3)} = \frac{1}{2}g^2vHW^+W^- + \frac{1}{2}\frac{g^2v}{2\cos^2\theta_w}HZZ$$
point interactions are
~~~
Coupling
$$\frac{1}{2}gv = \frac{2}{v}m_w^2$$

~~
$$\frac{1}{2}\frac{g^2}{\cos^2\theta_w} = \frac{2}{v}m_Z^2$$

Similarly, 4-pt interactions
$H^2W^+W^-$ *and* $H^2Z^2$
Give
~~
$$\frac{1}{2}g^2 = \frac{2}{v^2}m_W^2$$
~~
$$\frac{1}{2}\frac{g^2}{\cos^2\theta_w} = \frac{2}{v^2}m_Z^2$$

NOTE: it is obvious from this construction that the couplings of the higgs boson H to gauge bosons $W^+W^-Z$ are proportional to their masses (squared) !!
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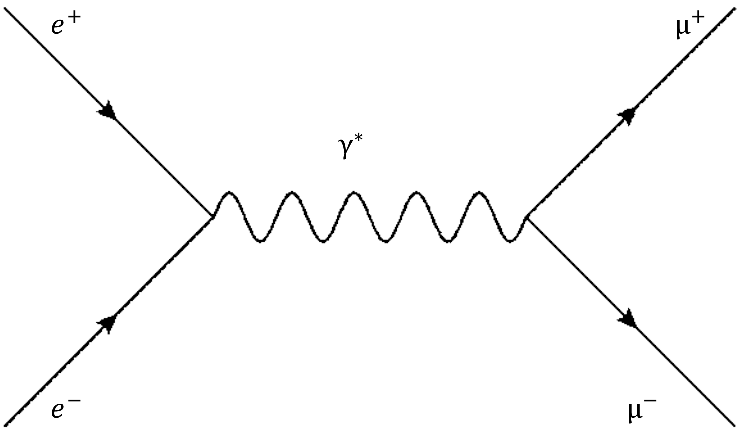
e^+
μ^+
γ^*
e^-
μ^-