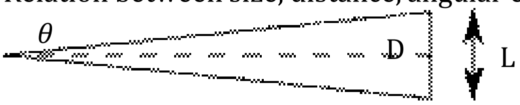
Problem sheet 1

21 February 2012
13:04

Q1: Parallax
    Relation between size, distance, angular extent



$D \gg L$
$\Rightarrow L \cong D\theta$

Q2: Relation between apparent + absolute magnitude

Q3: Binary system
    Parallax given -> distance
    Angular separation-> diameter of orbit
    Period of revolution given
        -> use kepler/newton
        Find total mass

Q4
    Cloud of gas of radius R
    Assume ρ constant (const density)
    Find gravitational potential

$$V(r) = -\frac{GM(r)}{r}$$

For r>R and r<R

Calculate total gravitational PE, Ω

$$\Omega = -\int_0^R \frac{GMdM}{r}$$

M(r)=cumulative
    Given constant ρ

Find total thermal energy
    Show that

$$U = \left(\frac{4}{3}\pi R^3 \rho\right)\frac{\frac{3}{2}kT}{m_p}$$

"Jeans Criterion"
    Cloud collapses on itself if $E = U + \Omega < 0$

$$\Rightarrow \rho > \frac{3}{4\pi M_{cloud}^2}\left(\frac{5kT}{2Gm_p}\right)^3$$

Jeans density

Assume $T \approx 10K$
    $M \approx M_\odot$
Find the radius if density is just = jeans density
Estimate total number of H atoms
Estimate Ω
Estimate "dynamical times"

Problem Sheet 2

22 March 2012
09:11

1.

i)

ii)

iii) Find total gravitational PE

$$\Omega = -\int_0^R \frac{Gmdm}{r}$$
$$= -\int_0^R \frac{Gm(r)}{r}\left(\frac{dm}{dr}\right)dr$$
$$= -\int_0^R G\frac{\frac{4}{3}\pi r^3\rho}{r}(4\pi r^2\rho)dr$$
$$= -\frac{16}{3}\pi^2 G\rho^2\int_0^R dr\, r^4$$
$$= -\frac{16}{15}\pi^2 G\rho^2 R^5$$

iv) Calculate U

Thermal energy of each H atom$=\frac{3}{2}kT$.

Number of H atoms

$$\frac{\text{Mass of star}}{\text{Mass of H atom}} = \frac{\frac{4}{3}\pi R^3\rho}{m_p}$$
$$U = \frac{\frac{4}{3}\pi R^3\rho}{m_p}\times\frac{3}{2}kT = \frac{2\pi R^3\rho}{m_p}kT$$

v) Jeans criterion

$U+\Omega<0$, system contracts

$$\frac{2\pi R^3\rho kT}{m_p} - \frac{16}{15}\pi^2 G\rho^2 R^5 < 0$$
$$\rho > \frac{3}{4\pi M_{cloud}^2}\left(\frac{5kT}{2Gm_p}\right)^3$$

Want to express R in terms of M_{cloud} & ρ

$$\frac{4}{3}\pi R^3\rho = M_{cloud}$$
$$\Rightarrow R = \left(\frac{3M_{cloud}}{4\pi\rho}\right)^{\frac{1}{3}}$$
$$\frac{3}{2}\frac{kT}{m_p}M_{cloud} < \frac{16}{15}\pi^2 G\rho^2\times\left(\frac{3M_{cloud}}{4\pi\rho}\right)^{\frac{5}{3}}$$
$$\left(\frac{3}{2}\frac{kT}{m_p}M_{cloud}\right)^3 < \left(\frac{16}{15}\pi^2 G\rho^2\right)^3\times\left(\frac{3M_{cloud}}{4\pi\rho}\right)^5$$
$$\Rightarrow \rho > \frac{3}{4\pi M_{cloud}}\left(\frac{\frac{5}{2}kT}{m_p G}\right)^3$$

vi) Plug in numbers

$T\approx 10K, M = M_0$

Find R for proto-star on the verge of collapse→ density=jeans density

$\rho = 1.77\times 10^{-15}kg/m^3$

$\Rightarrow R = 6.4 \rightarrow\times 10^{14}m$

$= 0.02\, parsecs$

vii) $\Omega \approx -2\times 10^{35}J$

$\# \text{ of atoms} \approx 1.2 \times 10^{57}$

viii) Dynamical time

$$t_{dyn} \approx \sqrt{\frac{R^3}{GM}}$$

$$= 1.4 \times 10^{12} s$$

2. 2

Example

A class of stars (called polytropes) have the equation of state

$$P = K\rho^n$$

Q1. Find the dependence of the mass of a polytrope, on its radius. (use scaling/dimensional arguments)

Q2. What happens when $n = \frac{4}{3}$ or $\frac{5}{3}$

Ans 1.

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

Dimensional analysis

$$\frac{P}{R} \sim -\frac{GM}{R^2} * \frac{M}{R^3}$$

$$\Rightarrow P \sim -\frac{GM^2}{R^4}$$

Apply

$$P = K\rho^n$$

$$k\rho^n \sim \frac{GM^2}{R^4}$$

$$k\left(\frac{M}{R^3}\right)^n \sim \frac{GM^2}{R^4} \Rightarrow kM^{n-2} \sim GR^{3n-4}$$

$$\Rightarrow M \propto R^{\frac{3n-4}{n-2}}$$

Ans 2.

When n=4/3 (relativistic fermi gas)

$$M \propto R^0$$

M=constant(Fixed universal value)

When n=5/3 (non-relativistic fermi gas)

$$M \propto R^{-3}$$

For n=5/3

$$M \propto \frac{1}{R^3}$$

Lesson: n=5/3 (non-relativistic white dwarf)

$$R \propto \frac{1}{M^{\frac{1}{3}}}$$

As R gets smaller, gas becomes relativistic

Solutions

30 April 2012
13:04

1.

Gravity "disappears"
Outward pressure causes gas to expand/explode
Apply newton's law to a shell of gas

$$m_{shell} \times \frac{d^2r}{dt^2} = P(r+dr)4\pi r^2 - P(r)\times 4\pi r^2 = < force\ on\ outer\ surface > - < force\ on\ inner\ surface >$$

$$\Rightarrow 4\pi r^2 dr \times \rho(r)\frac{d^2r}{dt^2} = \frac{dP}{dr}\times 4\pi r^2 dr$$

$$\frac{d^2r}{dt^2} = \frac{1}{\rho(r)}\frac{dP(r)}{dr}$$

Dimensional analysis

$$\frac{R}{t^2} \sim \frac{1}{\bar{\rho}}\frac{\bar{P}}{R}$$

$$\Rightarrow t \sim \frac{R}{\sqrt{\bar{P}/\bar{\rho}}} = \frac{R}{\text{speed of sound}}$$

$$\boxed{\text{Speed of sound in a gas} = \sqrt{\frac{\gamma P}{\rho}}}$$

2.

Time taken by a photon to emerge from a star

$R = 1.5R_\odot$

$l_{mfp} = 10^{-3}m$

Total distance travelled by photon after N scatterings

$= \sqrt{N} l_{mfp}$

$$1.5\,R_\odot = \sqrt{N} l_{mfp} \Rightarrow N = \left(\frac{1.5R_\odot}{l_{mfp}}\right)^2$$

Time for each step$= l_{mfp}/c$

$$= N \times \frac{l_{mfp}}{c} = \frac{(1.5R_\odot)^2}{l_{mfp}c} = 3.6\times 10^{12}s = 1.16\times 10^5 years$$

3.

Estimating P & T by dim analysis

$$P \sim \frac{GM^2}{R^4}$$

$$\left(\frac{dP}{dr} = \frac{GM\rho}{r^2}, \frac{dM}{dr} = 4\pi r^2 \rho(v)\right)$$

$M = 18M_\odot, R = 1100R_\odot$

$P = 2.46\times 10^5 N/m^2$

Estimating T (Virial theorem)

$Ke \sim Pe$

$$kT \sim \frac{GMm_{He}}{R} \Rightarrow T \sim \frac{GMm_{He}}{kR}$$

$= 1.5\times 10^6 K$

4.

$$\rho(r) = \rho_c\left(1 - \frac{r}{R}\right) \rightarrow given$$

(a) Find M(r)

$$\frac{dM}{dr}(r) = 4\pi r^2 \rho(r)$$

$$M(r) = \int_0^r 4\pi r^2\, \rho_c \left(1 - \frac{r}{R}\right) dr$$
$$= 4\pi\rho_c \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$
$$= \frac{4\pi r^3}{3} \rho_c \left(1 - \frac{3}{4}\frac{r}{R}\right)$$

(b)

Total mass,

$$M(r = R) = \frac{\pi r^3 \rho_c}{3}$$

(c)

Calculating pressure

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$
$$\Rightarrow \int_R^r dP = -\int_R^r \frac{GM(r)\rho(r)}{r^2} dr$$
$$P(r) - P(R)\Big|_{=0} = P(r) = -\int_R^r \frac{GM(r)\rho(r)}{r^2} dr$$
$$= -\int_R^r G \frac{4}{3}\pi\rho_c r \left(1 - \frac{3}{4}\frac{r}{R}\right) \times \rho_c \left(1 - \frac{r}{R}\right)$$

Define $x = r/R$

$$= -\frac{4}{3}\pi G \rho_c^2 R^2 \int_1^{\frac{r}{R}} x \left(1 - \frac{3x}{4}\right)(1 - x) dx$$
$$\rightarrow Algebra$$
$$P(r) = \frac{4}{3}\pi G\, \rho_c^2 R^2 \left(\frac{5}{48} - \left(\frac{r^2}{2R^2} - \frac{7}{12}\frac{r^3}{R^3} + \frac{3}{16}\frac{r^4}{R^4}\right)\right)$$

5.

Given core temperature + radius of star, estimate temp gradient

$$\frac{dT}{dr} \sim \frac{\Delta T}{\Delta r} = \frac{T_{surface} - T_{core}}{R} = -\frac{T_{centre}}{R} = -\frac{1.5 \times 10^7 K}{7 \times 10^8 m} = -0.02$$

(b) Ratio of energy flux at centre and surface

$$\frac{\text{Flux at centre}}{\text{Flux at surface}} = \frac{T_{centre}^4}{T_{surface}^4} = \frac{(1.5 \times 10^7 K)^4}{(5770K)^4} = 5 \times 10^{13}$$

6.

Given ρ for sun, estimate distance between 2 e^-'s in core plasma

$\rho_c = 1.5 \times 10^5 kg/m^3$

Total no of H atoms/m^3

$$= \frac{\rho_c}{m_H} = \frac{\left(1.5 \times \frac{10^5 kg}{m^3}\right)}{1.67 \times 10^{-27} kg} = 9 \times 10^{31}\ /m^3$$

1 e^- per atom

$n_e = 9 \times 10^3\ /m^3$

Typical separation

$$= \frac{1}{n_e^{\frac{1}{3}}} = 2.2 \times 10^{-11} m$$

(b) Estimate coulomb potential between $2e^-$'s

$$= \frac{e^2}{4\pi\epsilon_0 r} = 64.5 eV$$

7.

Determine whether electron gas in a star with $T = 10^8 k, and\ n_e = 3 \times 10^{34}/m^3$ is degenerate or not

Matter is degenerate if

$$\text{Distance between } e^- < \frac{1}{2} \text{thermal wavelength f } e^-{'s}$$

$$n_e^{-1/3} < \frac{h}{2\sqrt{3m_e kT}}$$

$$\sqrt{3m_e kT} \rightarrow \left(\frac{p^2}{2m_e} = \frac{3}{2}kT \Rightarrow p = \sqrt{3m_e kT}\right)$$

$$n_e^{-\frac{1}{3}} = 3.2 \times 10^{-12} m$$

$$\frac{h}{2\sqrt{3m_e kT}} = 5.4 \times 10^{-12} m$$

⇒ electron gas is a degenerate fermi gas

Equations

13 March 2012
13:03

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Need equation of state (e.g. $P = k\rho T$)

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dL}{dr} = 4\pi^2 \rho(r)\mathrm{E(r)}$$

$$\frac{dT}{dr} \rightarrow L$$

Radiative transport

Foundations of Astrophysics

30 January 2012
13:04

Prem Kumar

1. Astrophysics in a nutshell- Dan Maoz (Princeton U. press)
2. (Advanced)- An intro to Modern Astrophysics- Carroll and Osthe
3. Intro to stellar structure by S. Chandrasekhar

This course: mainly about stellar structure
Outline:

1. Review observational results
2. Black body radiation, Stellar parameters, HR diagram
3. Stellar Models
4. Equations of stellar structure
5. Stellar nucleon reactions
6. Features of solutions to equations
7. Stellar evolution, remnants

Length scale
Daily life $\sim 10^0 m - 10^3 m$

Radius of earth$\sim 10^7\ m$
Radius of Solar system $\sim 10^{11} m$
$R_{\odot} \sim 7 * 10^8 m$
$R_{milky\ way} \sim 10^5 ly \sim 10^{21} m$
$R_{observable\ universe} \sim 10^{10} ly$
Mass scale of a star
$M_{\odot} \sim 2 * 10^{30} kg$
<u>Example:</u>

Estimate the typical size of an atom

Solve Schrodinger eq for electron

$$Ke + Pe = total\ energy$$

$$Ke = \frac{p^2}{2m_e}$$

$$p = i\hbar \frac{d}{dx}$$

$$Pe = -\frac{e^2}{4\pi\epsilon_0 r}$$

$$Ke \sim Pe$$

$$\frac{1}{m_e}\left(\hbar \frac{d}{dx}\right)^2 \sim \frac{e^2}{4\pi\epsilon_0 r}$$

$$-\frac{\hbar^2}{m_e}\frac{1}{a^2} \sim \frac{e^2}{4\pi\epsilon_0 a}$$

$$a = \frac{\hbar^2}{m_e\left(\frac{e^2}{4\pi\epsilon_0}\right)}$$

$$\sim 10^{-9} - 10^{-11}$$

Astrophysics is applied physics

Classical mech
Thermodynamics
Statistical physics
Electromagnetism
Quantum mechanics
Nuclear physics

Stars/stellar structure: Observations

30 January 2012
13:40

Sun-> best studied star

- Solar interior is not observable
  - Gas is ionized
- Neutrinos (light, weakly interacting particles)
  - Nuclear processes in solar core
- Quakes on the sun's surface- helioseismology
  - Indirect source of information on solar core
- Want to construct stellar models that reproduce observed properties of sun/other stars

Basic observable parameters of stellar physics

1. Mass "M"
2. Luminosity=L=Total energy radiated per second (Watts)

   $$L = \int_0^\infty d\nu L(\nu)$$

   $$L(\nu) = power\ emitted\ at\ frequency\ \nu$$
3. Radius "R"
4. T_{eff} →Effective temperature

   =Temperature associated to a black body of the same radius as the star, and radiating energy at the same rate as the star

   <u>Stefan's Law:</u> Total energy radiated per second by a black body

   $$L = \underbrace{4\pi R^2}_{Surface\ Area} \underbrace{\sigma}_{Stefan's\ constant} T_{eff}^4$$

   $$T_{eff} = \left(\frac{L}{4\pi R^2 \sigma}\right)^{\frac{1}{4}}$$

   $$\sigma = 5.67 \times 10^{-8}\ Wm^{-2}K^{-4}$$

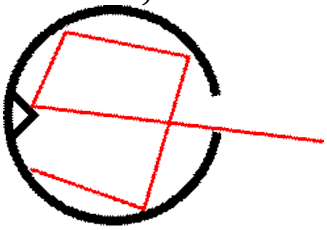
M, L, R, T_{eff}

Only 3 are independent observables

Detour: Black bodies Review

31 January 2012
13:18

A black body is an object that absorbs (when "cold") all radiation incident on it (reflects or emits none)

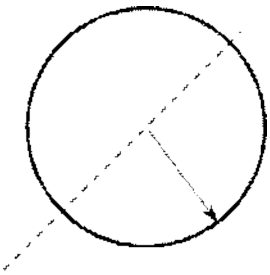


Radiation incident on aperture is absorbed
Eventually system equilibrates

Intensity is vanishingly small at high frequencies

How do we obtain energy density/frequency?

Use statistical mechanic.
How many photons are there per unit volume with momentum between p and p+dp?
In phase space



$$\frac{\#\,of\,photons}{volume} = \frac{1}{h^3} 4\pi p^2 dp \times 2 \times n(p)$$

2->2 possible polarisations

n(p)->occupation number->bose-einstein distribution

$$= \frac{8\pi p^2 dp}{h^3} n(p)$$

For photons

$$p = \frac{h\nu}{c}$$

$$E = energy = h\nu$$

$$\frac{\#\,of\,photons}{volume} = \frac{8\pi \nu^2 dV}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$\frac{1}{e^{\frac{h\nu}{kT}} - 1} \rightarrow$$

guessed by Planck to fit observed data

$$\boxed{\frac{8\pi \nu^2 d\nu}{c^3} \times \frac{hV}{e^{\frac{h\nu}{kT}} - 1} = u(\nu) d\nu}$$

Before Planck, people thought classically, the average energy per degree of freedom =kT by equipartition theorem

$$u(\nu) d\nu = \frac{8\pi \nu^2 d\nu}{c^3} kT$$

Increases infinitely- UV catastrophe

Look at
$$\frac{h\nu}{e^{\frac{h\nu}{kT}}-1}$$
$e^x \approx 1+x$

For $x \ll 1$

Low ν: $\frac{h\nu}{kT} \ll 1$
$$\frac{h\nu}{1+\frac{h\nu}{kT}-1} = kT$$
"classical"

High ν
$$\frac{h\nu}{e^{\frac{h\nu}{kt}}-1} \to h\nu e^{-\frac{h\nu}{kT}}$$
$e^{-k} \ll 1\ for\ x \gg 1$

Quantum mechanics prevents uv catastrophe

Blackbody radiation

$u(\nu)d\nu$
$$= \frac{8\pi\nu^2}{c^3}\ d\nu \frac{h\nu}{e^{\frac{h\nu}{kT}}-1}$$
$$u(\nu) \propto \frac{\nu^3}{e^{\frac{h\nu}{kT}}-1}$$
Near $\nu = 0$
$$u(\nu) \propto \frac{\nu^3}{1+\frac{h\nu}{kT}-1} = \frac{\nu^2 kT}{h}$$
$$\propto \nu^2$$
For ν large
$$u(\nu) \propto \nu^3 e^{-\frac{h\nu}{kT}} \ll 1$$
Exercise: find condition for $u(\nu)$ to attain its maximum
$$\nu_{max} \propto \frac{kT}{h} \#$$
$$\nu_{max} \approx \frac{2.82kT}{h}$$
e.g. for the sun surface temperature (T_{eff}) ~$5800K$

Wavelength $\left(\lambda_{max} = \frac{c}{\nu_{max}}\right) \approx 5000$ Å→ $green\ light$

What is the total energy density of the radiation across all frequencies?
$$U_{total} = \int_0^\infty U(\nu)d\nu$$
$$= \int_0^\infty \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}}-1}d\nu$$
Define $x \equiv \frac{h\nu}{kT}$
$$dx = \frac{h}{kT}d\nu$$
$$= \frac{8\pi}{c^3}\frac{kT}{h}\ kT\frac{(kT)^2}{h^2}\int_0^\infty \frac{dx * x}{e^x - 1}$$
$$= \frac{8\pi k^4 T^4}{c^3 h^3}\int_0^\infty \frac{dx * x}{e^x - 1}$$
$$= \frac{8\pi k^4 T^4}{c^3 h^3}\frac{\pi^4}{15}$$
$$\boxed{U_{tot} = aT^4}$$
$a = radiation\ constant$

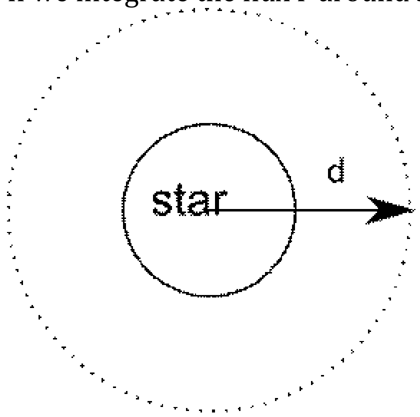
$$= \frac{8\pi^5 k^4 T^4}{15 c^3 h^3}$$

Corollary: total flux emitted at the surface of a blackbody radiator $F = \sigma T^4$

$$\sigma = \frac{ac}{4} = stefan's\ constant$$

Flux=total energy emitted/unit time/unit area (perpendicular to direction of emission)

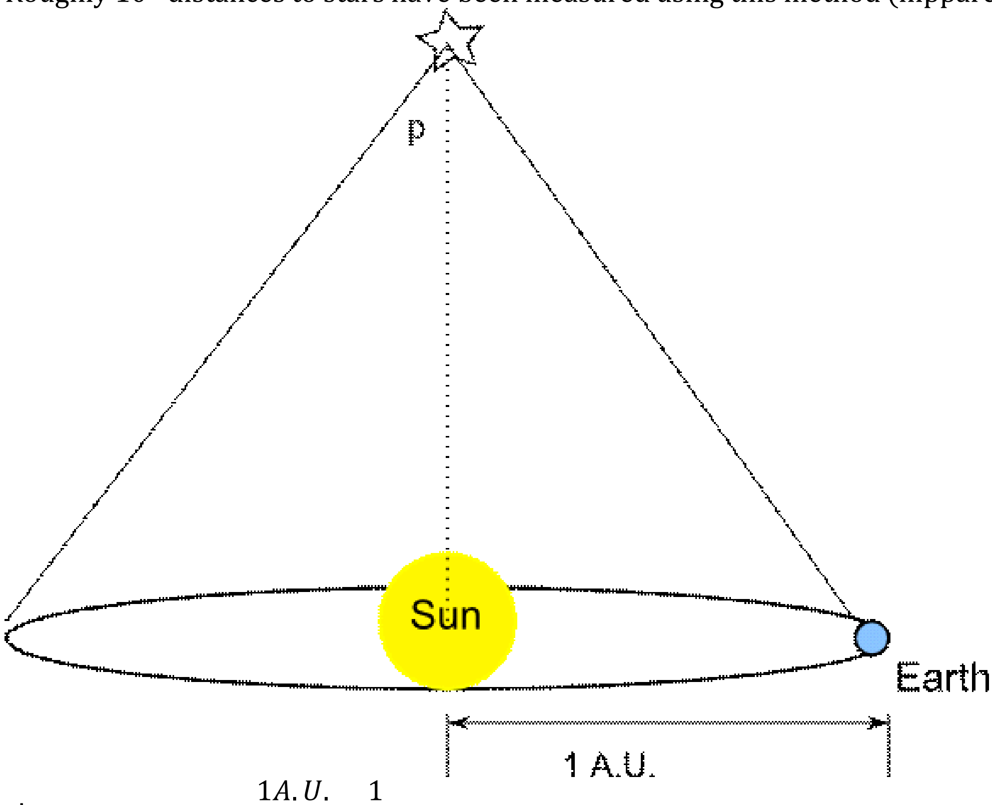
If we integrate the flux F around sphere of radius d, we get luminosity "L" of star: $\boxed{L = 4\pi d^2 F}$



Back to astro

06 February 2012
13:32

- How do we measure distances to stars?
  Parallax: apparent stellar motion due to orbit of earth around the sun
  Roughly 10^5 distances to stars have been measured using this method (hipparcos)



$$\sin p \approx \tan p \approx p = \frac{1 A.U.}{d} = \frac{1}{p} parsecs$$

$$1\ parsec = \frac{1\ A.U.}{1 arcsecond} = \frac{1.5 \times 10^{11} m}{\left(\frac{2\pi}{360 \times 60 \times 60}\right)} = 3.09 \times 10^{16} m \approx 3.27\ lightyears$$

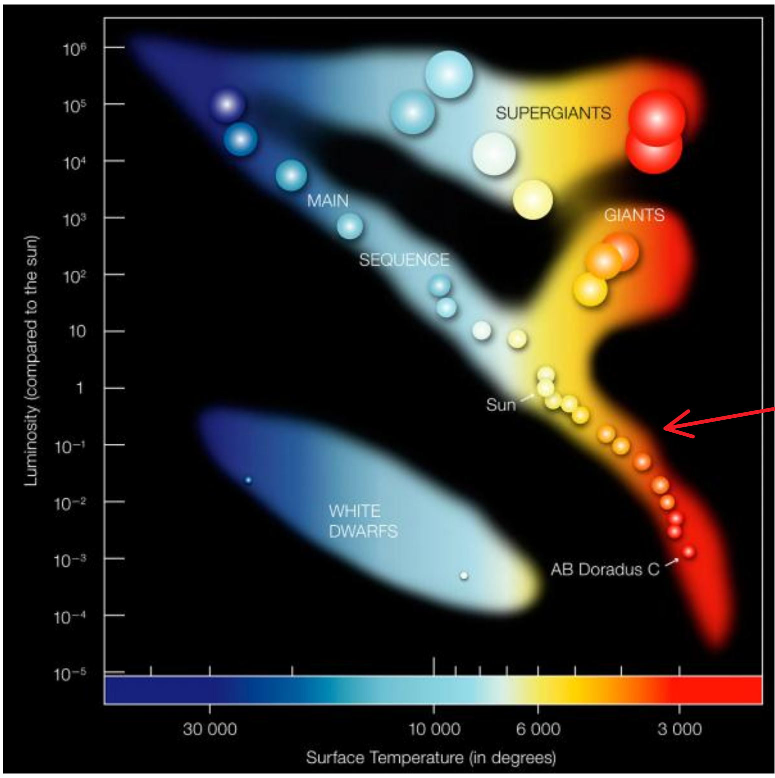
Stellar radius measurement
$R_{\odot} \approx 7 \times 10^5 km$
Extremely difficult to measure stellar radii
→ interferometry

Summary of radius measurements

- "Main sequence" stars have radii similar to sun. "R" increase slowly as a function of surface temperature
- A few stars have $R \sim 0.01 R_{\odot}$
  (white/brown dwarfs)
- A few stars have $R > 10 R_{\odot}$
- In all
  $0.01 R_{\odot} < R < 100 R_{\odot}$

H-R Diagram

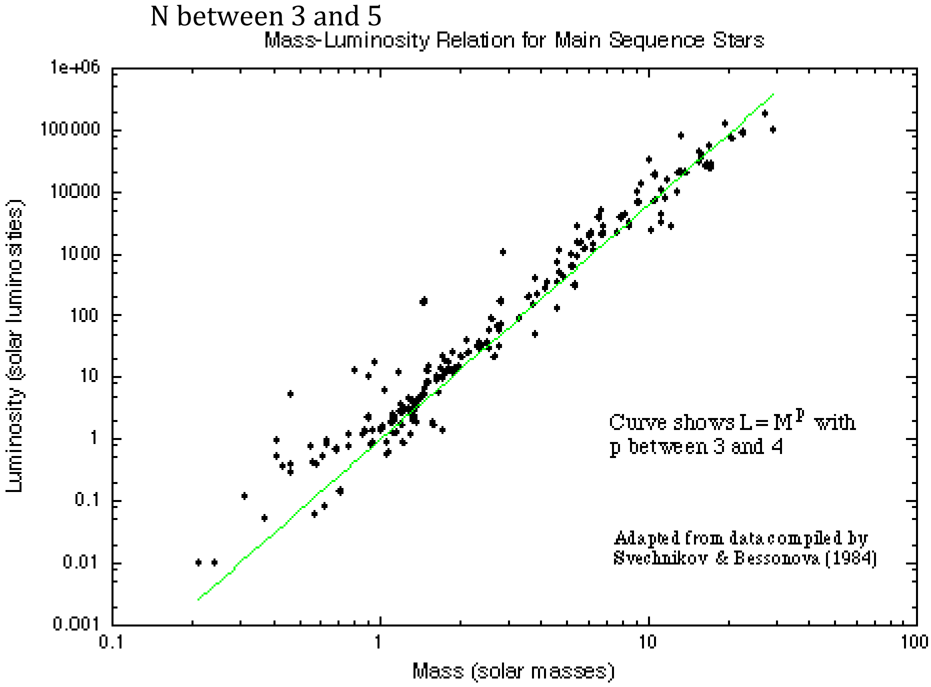
07 February 2012
13:04



Band of "Main Sequence" stars

Any stellar model must explain basic features of the H-R diagram

Mass-Luminosity relation
For the main sequence stars whose masses have been measured
Empirical observation
$L \propto M^n$
N between 3 and 5



Stellar model should explain this.
Luminosity measured by measuring the flux F from a source
$\Rightarrow \boxed{4\pi d^2 F}$

$$F = \frac{L}{4\pi d^2}$$

For astronomical measurements, the brightness (flux) is denoted by "magnitude":

Apparent magnitude

$$m \propto \log_{10} F$$

Relative magnitude

$$\boxed{m_1 - m_2 = -2.5 \log_{10}\left(\frac{F_2}{F_1}\right)}$$

For reference, VEGA constellation was chosen to have magnitude=0

With this choice,

$$M_{\odot} = -26.8$$

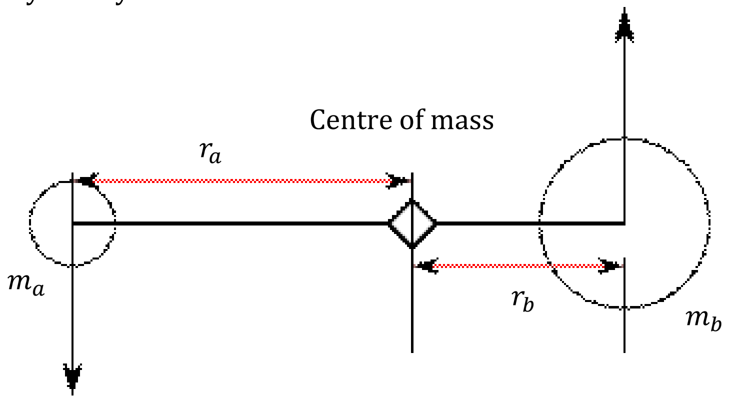
Absolute magnitude

Apparent magnitude of the object when placed at a distance of 10 parsecs

$$\boxed{m - m_{abs} = 5 \log\left(\frac{d}{10}\right)}$$

How to measure masses of stars?

- Binary star systems → useful for mass measurements



Assuming for simplicity that orbits are circular,

If the angular velocity of rotation is "ω" $= \frac{2\pi}{T}$

$$m_a \omega^2 r_a = \frac{G m_a m_b}{(r_a + r_b)^2} = m_b \omega^2 r_b$$

Eq 1

We also have the relation

$$m_a r_a = m_b r_b$$

Eq 2

1&2

$$\Rightarrow \omega^2 = \frac{G(m_b + m_a)}{(r_a + r_b)^3}$$

Suppose T is observable

If T is known & r_a, r_b can be measured, then the masses m_a, m_b can be inferred

Upshot of mass measurements most stars have

$$0.1 M_{\odot} < M < 200 M_{\odot}$$

$m_1 \& m_2 \rightarrow 2\ masses$

Force of gravity

$$= \frac{G m_1 m_2}{r^2}$$

Potential energy

$$= -\frac{G m_1 m_2}{r}$$

(with reference to a point at ∞)

Our formula "mgh" is PE measured at height h from earth's surface- rather than reference at point at ∞

What is the exact PE of mass "m" in the gravitational field of the earth?

$$PE = -\frac{GmM}{h+R} - \left(-\frac{GMm}{R}\right)$$

M=earth mass

m=object mass

h=height from surface

R= earth's radius

$$\approx \left(\frac{GM}{R^2}\right)(mh)$$

Since $h \ll R$

$\frac{GM}{R^2} \rightarrow$"g"

$\Rightarrow mgh$

Gravitational potential

$$V = -\frac{Gm_1 m_2}{r}$$

Potential energy w.r.t. point at ∞

What is the solar mass?

Apply Newton's law

$$\frac{GM_\odot m_{earth}}{r^2} = m_{earth}\omega^2 r$$

$$\Rightarrow M_\odot = \frac{4\pi^2 r^3}{T^2 G}$$

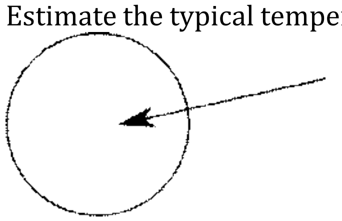
$$M_\odot = \frac{4\pi^2(1.5\times 10^{11} m)^3}{(86400s \times 365)^2 \times 6.67\times 10^{-11}} \cong 2\times 10^{30} kg$$

Estimate the gravitational PE of the sun

$$V \sim -\frac{GM_\odot^2}{R_\odot} = ?$$

Exercise

Estimate the typical temperature inside the sun



Typical kinetic energy

$\sim kT$

Thermal energy

$$PE \sim -\frac{GM_\odot m_H}{R_\odot}$$

Ke & Pe are comparable

$$\frac{GM_\odot m_H}{R_\odot} \sim kT$$

$$T \sim \frac{GM_\odot m_h}{R_\odot k}$$

$$T \sim \frac{10^{11}\times 10^{30} kg \times 10^{-27} kg}{10^8 m \times 10^{-23}\frac{J}{k}}$$

$\boxed{T \sim 10^7 K}$ ⇒Gas in sun is ionized ⇒ Plasma

Plasma is a better ideal gas than neutral H gas

Size of H nucleus (proton) $\sim 10^{-15} m \ll 10^{-10} m$ (size of H atom)

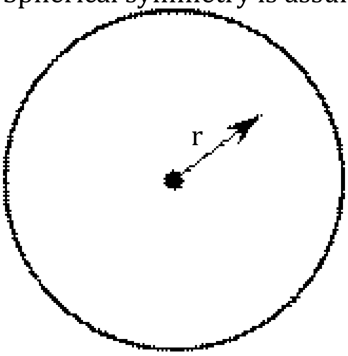
Stellar Structure Models

13 February 2012
13:28

Star: Ball of gas held together by gravity.
Internal pressure balances gravity

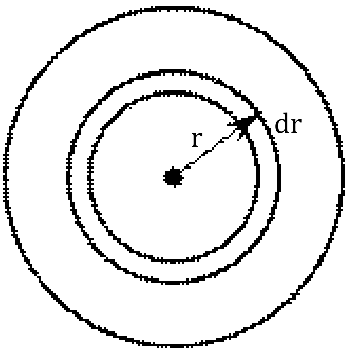
- Gravity/Pressure equilibrium
  Equation of hydrostatic equilibrium
- Mass-density relation (mass continuity equation)
- Conservation of energy
  Star radiates energy⇒ For stability, we need a constant source of energy production
- Transport of energy from interior to surface
  - Supplementary equation →
    - equation of state (ideal gas law, or more complicated e.g. fermi degeneracy pressure)
    - Opacity (how opaque the gas is to radiation)
    - Core energy production rate

1. Hydrostatic support eqn
   Spherical symmetry is assumed



⇒All properties are functions of "r"→ distance from origin
Let density at distance "r" be $\rho(r)$

Pressure	$P(r)$
Temperature	$T(r)$
Cumulative mass	$M(r)$



Thin shell is in equilibrium
Inward force=outward force
Outward force=force exerted by gas on the inner surface of the shell
$P(r) \times 4\pi r^2$
Inward force: pressure of gas on outer surface + force of gravity

$$= P(r+dr) \times 4\pi r^2 + \frac{G\left(4\pi r^2 dr \times \rho(r)\right) \times M(r)}{r^2}$$

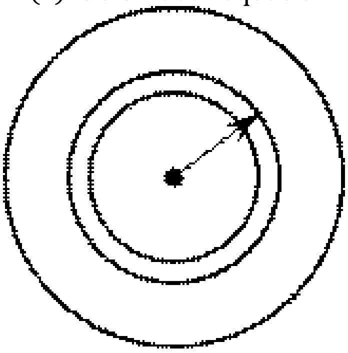
$$\Rightarrow P(r) = P(r+dr) + \frac{GM(r)dr\rho(r)}{r^2}$$

$$\Rightarrow P(r) - P(r+dr) = \frac{GM(r)\rho(r)dr}{r^2}$$

$$\Rightarrow \lim_{dr\to 0}\frac{P(r)-P(r+dr)}{dr}=\frac{GM(r)\rho(r)}{r^2}$$

$$\Rightarrow \boxed{-\frac{dP}{dr}(r)=\frac{GM(r)\rho(r)}{r^2}}$$

$M(r)$ is a smooth equation



Mass of shell=volume of shell * density function

$= 4\pi r^2 dr\ \rho(r)$

But

$M(r+dr)-M(r)=4\pi r^2 dr\ \rho(r)$

$$\lim_{dr\to 0}\frac{M(r+dr)-M(r)}{dr}=4\pi r^2\rho(r)$$

$$\Rightarrow \boxed{\frac{dM(r)}{dr}=4\pi r^2\rho(r)}$$

Have estimated

$T_\odot \sim 10^7 k$

$M_\odot \sim 10^{30} kg$

$\rho_\odot \sim ?$

$P_\odot \sim$

Density estimate

$$\bar{\rho}_\odot=\frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}=\frac{2\times 10^{30} kg}{\frac{4}{3}\pi(7\times 10^8)^3}$$

$= 1390\ kg/m^3$ (density of $H_2 0 \sim 1000\ kg/m^3$)

$\bar{\rho}_\odot$ =average density

Pressure estimate

$$\frac{dP}{dr}=-\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{P_\odot}{R_\odot}\sim\frac{GM_\odot}{R_\odot^2}\times\frac{M_\odot}{R_\odot^3}$$

$$P_\odot\sim\frac{GM_\odot^2}{R_\odot^4}$$

$$=\frac{\left((6.67\times 10^{-11})\times(10^{30} kg)\right)^2}{(7\times 10^8 m)^4}\cong 10^{15}\ N/m^2 \sim 10^{10} atm$$

<u>Minimum possible value of central pressure</u>

$$\frac{dP}{dr}=-\frac{GM\rho}{r^2}$$

(1)

$$\frac{dM}{dr}=4\pi r^2\rho$$

(2)

$$\frac{dP/dr}{dM/dr}=\boxed{\frac{dP}{dM}=-\frac{GM}{4\pi r^4}}$$

Equation independent of $\rho(r)$

$$dP=-\frac{gM}{4\pi r^4}dM$$

Integrate

$$\int_{centre}^{surface} dP = -\int_{centre}^{surface} \frac{GM}{4\pi r^4} dM$$

$L.H.S. \Rightarrow P_{surface} - P_{centre}$

$= 0 - P_{centre}$

$$P_{centre} = \int_{centre}^{surface} \frac{G}{4\pi}\frac{MdM}{r^4}$$

Can't do this integral, but can put a bound on it.

$r(M) < R$

R=stellar radius

$\Rightarrow \frac{1}{r} > \frac{1}{R}$

$\Rightarrow \frac{1}{r^4} > \frac{1}{R^4}$

$\rightarrow \frac{GM}{r^4} > \frac{GM}{R^4}$

$$P_{centre} > \int_{centre}^{surface} \frac{GMdM}{4\pi R^4} = \frac{G}{4\pi R^4}\int_{centre}^{surface} MdM = \frac{G}{4\pi R^4}\left(\frac{M^2}{2}\Big|_{surface} - \frac{M^2}{2}\Big|_{centre}\right)$$

$= \frac{G}{8\pi R^4} M_*^2$

M_*^2 =mass of the star

$$\boxed{P_{centre} > \frac{GM_*^2}{8\pi R^4}}$$

For sun

$4.5 \times 10^{13}\ N/m^2 \approx 4.4 \times 10^8 atm$

2. Viral theorem

Relates thermal energy of star to gravitational PE

$\Omega = -2\,U$

Ω =gravitational PE

U=thermal energy

Eq of stellar structure

$\Rightarrow \frac{dP}{dM} = -\frac{GM}{4\pi r^4}$

Multiply both sides by $4\pi r^3$

$4\pi r^3 \frac{dP}{dM} = -\frac{GM}{r}$

$4\pi r^3 = 3 \times$ volume of region enclosed by shell of radius r

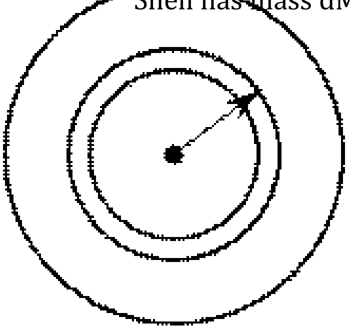
$= 3 \times \left(\frac{4}{3}\pi r^3\right) = 3V(r)$

$\Rightarrow 3V\frac{dP}{dM} = -\frac{GM}{r}$

$\rightarrow 4V\ dP = -\frac{GM}{r} dM$

$$3\int_{centre}^{surface} VdP = -\int_{centre}^{surface} \frac{GMdM}{r}$$

Shell has mass dM



$L.H.S. \int_c^s VdP$

$$= \int_c^s d(PV) - \int_c^s PdV$$

$$= PV\Big|_{surface} - PV\Big|_{centre} - \int_c^s PdV$$

Pressure at surface=0

Volume at centre=0

$$= 0 - 0 - \int_c^s PdV$$

$$\boxed{3\int_c^s PdV + \Omega = 0}$$

$\Omega = \text{negative}$

Apply to ideal gas

For ideal <u>non-relativistic</u> gas, $PV = NkT$

$3\int (n(dV)kT) + \Omega = 0$

$\Rightarrow PV = \frac{N}{V}kT$

n=number density

$\int P\, dV = \frac{2}{3}\int \left(\frac{3}{2}kT\, n\right) dV$

$\left(\frac{3}{2}kT\, n\right)$=thermal energy per unit volume

$= \frac{2}{3} \times U$

U=total thermal energy

$3 * \frac{2}{3} * U - 2U = 0$

<u>Minimum mean temperature</u>

Gravitational PE

$$\Omega = -\int_0^{surface} \frac{GMdM}{r}$$

(M=cumulative mass at distance r from centre)

Since $r < R$

$$\frac{1}{r} > \frac{1}{R}$$

$$\int_0^{Surface} \frac{GMdM}{r} > \int_0^{Surface} \frac{GMdM}{R} = \frac{GM_*^2}{2R}$$

$$|\Omega| > \frac{\mathrm{GM}_*^2}{2\mathrm{R}}$$

From virial theorem

$3\int PdV + \Omega = 0$

$\Rightarrow -\Omega = 3\int PdV = 3\int P\frac{dM}{\rho}$

$\uparrow$ *because* $\rho = \frac{dM}{dV}$

(No assumptions about P,ρ, M)

$P = P_{gas} + P_{radiation}$

$P_{gas} = gaseous/kinetic\ pressure$

Assume $P_{radiation} \ll P_{gas}$

Assumptions

- Pressure dominated by matter
- Take gas to be ideal

Pv=NkT

$\Rightarrow P = \frac{N}{V}kT$

$P = nkT$

n=number density

Let the average mass of each particle of ideal gas be m

Density $\rho = n \times m$

$$-\Omega = 3\int P\frac{dM}{\rho}$$
$$= 3\int nkT\frac{dM}{\rho}$$
k, m constant

Now we use
$$-\Omega > \frac{GM_*^2}{2R}$$
$$\Rightarrow \frac{3k}{m}\int TdM > \frac{GM_*^2}{2R}$$
$$\Rightarrow \int TdM > GM_*^2 \times \frac{m}{6kR}$$
$$\Rightarrow \frac{\int TdM}{M_*} > \frac{GM_*m}{6kR}$$
$\bar{T}$ (average temperature
$$\bar{T} > \frac{GM_*m}{6k}$$
Recall the estimate

KE of any particle of ideal gas ~kT

|PE of this particle|~ $\frac{GmM_*}{R}$

By Virial thm
$$\frac{GmM_*}{R} \sim kT$$
$$\Rightarrow T \sim \frac{GmM_*}{kR}$$
By dimensional analysis- out by order of 6- very useful for estimations in astro

Take m=proton mass=$1.67 \times 10^{-27} kg$

$M_* = M_\odot = 2 \times 10^{30} kg$

$R = R_\odot$

$\bar{T}_\odot > 4 \times 10^6 k$

Demonstrates that gas inside the sun is ionized

Ionization energy of H atom

~10eV

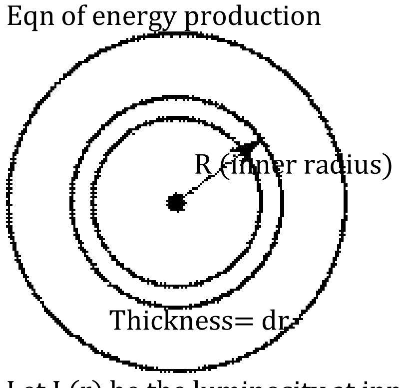
$= 1.6 \times 10^{-18} J$

↑ionization temperature $kT = 1.6 \times 10^{-18} J$
$$T = \frac{1.6 \times 10^{-18} J}{1.38 \times 10^{-23}\frac{J}{K}} \cong 10^5 k$$
Justifies ideal gas assumption

Atoms = plasma

3. Conservation of Energy

Eqn of energy production



Let L(r) be the luminosity at inner surface of the shell.

L(r+dr)=luminosity at outer surface of the shell

Let $\epsilon(r)$ =energy released by unit mass of the shell per unit time (due to fusion)

Mass of shell= $4\pi r^2 dr \times \rho(r)$

Total energy generated by the shell/unit time = $\epsilon(r) \times 4\pi r^2 dr\rho(r)$

Conservation of energy
$\Rightarrow L(r+dr) - L(r) = 4\pi r^2 \rho(r)\epsilon(r)dr$
In the limit that $dr \to 0$

$$\boxed{\frac{dL}{dr} = 4\pi r^2 \rho(r)\epsilon(r)}$$

So far we have
$P(r), M(r), L(r), \rho(r), \epsilon(r)$
5 unknowns & 3 equations

Energy transport

If energy is produced in stellar intension, how is it transported,

- Allows to determine T(r)

Modes of energy transport

Convection, conduction, radiation
Radiation is most important for most stars
How does radiation transport energy?
Photons transfer energy by scattering
→ extremely important in plasma
Two types of scattering
Thompson scattering
Compton scattering

Some consequences of violation of hydrostatic equilibrium.

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Two extreme situations
Pressure is $\gg$ force of gravity
Star to expand explosively → what is the timescale for this?

Pressure is $\ll$ force of gravity
Star collapses → what is the timescale for this?
↑Dynamical time

Estimating Dynamical time

Every shell in the cloud experiences free fall under gravity in absence of pressure
Apply newton's law to each shell
In particular, outermost shell : find time to collapse
Apply to outermost shell (surface) at R(t)

$$m_{shell}\frac{d^2R}{dt^2} = -\frac{GM^* m_{shell}}{R^2}$$

$$\frac{d^2R}{dt^2} = -\frac{GM^*}{R^2}$$

Estimated timescale for collapse (t_{dyn})

$$\frac{R}{t_{dyn}^2} \sim \frac{GM^*}{R^2}$$

$$t_{dyn} \sim \sqrt{\frac{R^3}{GM^*}}$$

$\to_{\text{for sun}}$ $1600 sec$
When pressure $\gg$ gravity
→ stars explode
→ Use P, ρ, R to form a quantity with dimensions of time
Time taken by sound to propagate from centre to surface
Equation of energy production

$$\frac{DL'}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

$\epsilon(r)$ rate of energy production per unit mass
Estimate $\epsilon(r)$ for the sun

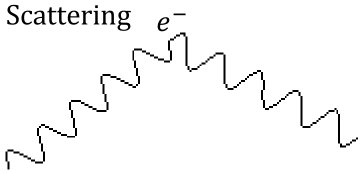
$$\frac{L_\odot}{\dot{R}} \sim R_\odot \rho_{\odot\epsilon} \Rightarrow \epsilon \sim \frac{L_\odot}{R_\odot^2 \rho_\odot} = \frac{L_\odot}{M_\odot}$$
$$L_\odot = 4 \times 10^{26} W$$
$$M_\odot = 2 \times 10^{30} kg$$
$$\frac{L_\odot}{M_\odot} = 2 \times 10^{-4} W/kg$$

Energy transport

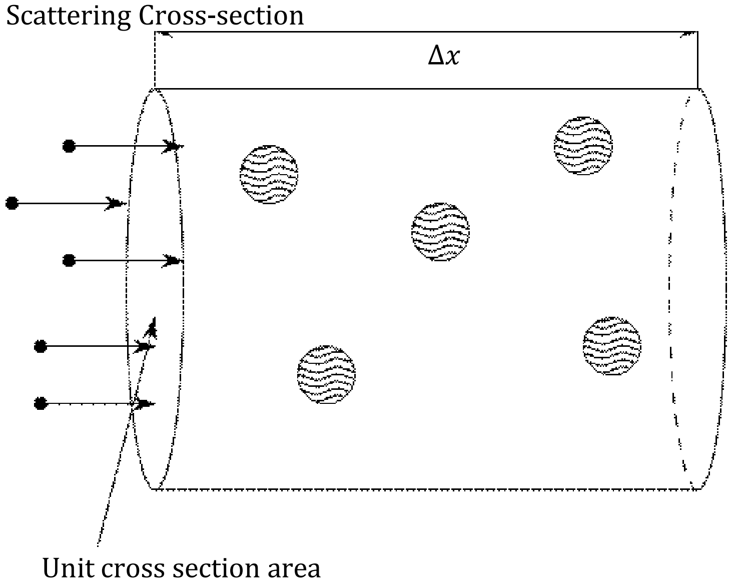
Radiative plasma

↓

Scattering



Scattering Cross-section



$$\frac{\text{Mumber of scatterings}}{\text{Number of incident particles/photons}}$$
$$= \frac{\text{Number of obstacles} \times \text{effective cross section of each obsticle}}{\text{Cross section of tube}}$$

Number of scattering processes
= Number of obstacles × Effective cross-section × Flux of incoming particles (photons)

Suppose n=number density of charged particles. (Obstacles)

Number of scattering centres/charges
$= n \times$ volume of tube
$= n\Delta x$

Because cross section=unit area ($1\ m^2$)

Let σ be the effective cross section of each scattering centre/charge

$$\text{Fraction of area covered by targets} = \sigma n \Delta x = \frac{\#\text{ of scattering}}{\#\text{ of incoming photons}}$$

Take 1 incident photon,

Define the distance traversed by a photon in between two successive scatterings as mean free path l_{mfp}

So,

$$\sigma n l_{mfp} = 1$$
$$\boxed{l_{mfp} = \frac{1}{\sigma n}}$$

If stellar matter has a complex chemical composition (varies nuclei)

(n_i, σ_i)

n=number density (units of length^-3)

σ= effective cross section (units of length^2)

$$\left(\sum_i \sigma_i n_i\right) \times l_{mfp} = 1$$

$$\boxed{l_{mfp} = \frac{1}{\sum_i \sigma_i n_i}}$$

$$l_{mfp} = \frac{1}{\kappa\rho}$$

$\kappa = opacity$

$\rho = density$

$\kappa = \kappa(T, \rho, \text{chemical composition})$

Estimate of l_{mfp} from scattering cross-section

e.g. suppose that photons undergo Thomson scattering

$$\sigma_{thomson} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.7 \times 10^{-29}\, m^2$$

Assume that all stellar matter is H

$$n_e = \frac{\rho}{m_H}$$

$\rho \rightarrow use\ average\ density$

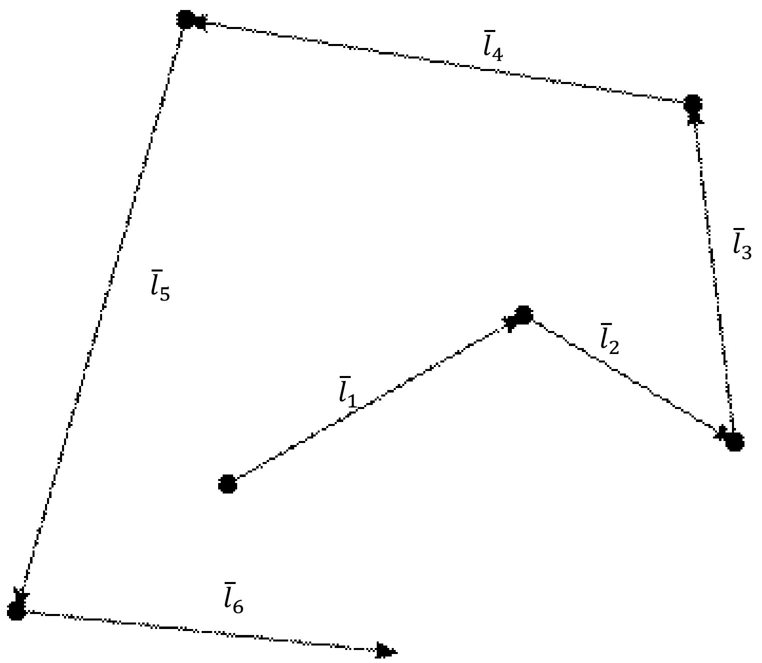
$$l_{mfp} = \frac{1}{n_e \sigma_{thomson}}$$

$\cong 2 \times 10^{-2} m$

$\cong 2cm \rightarrow$ overestimate for central regions of the star where ρ is much higher

In core region $l_{mfp} \approx 10^{-3} m$

- Photons carry away energy by a Random Walk



Total displacement of photon after N scatterings

$\bar{D} = \bar{l}_1 + \bar{l}_2 + \bar{l}_3 + \cdots + \bar{l}_N$

Average displacement $< \bar{D} > \ = 0$

$\bar{D} * \bar{D} = (\bar{l}_1 + \bar{l}_2 + \bar{l}_3 + \cdots + \bar{l}_N)(\bar{l}_1 + \bar{l}_2 + \bar{l}_3 + \cdots + \bar{l}_N)$

$= \bar{l}_1^2 + \bar{l}_2^2 + \bar{l}_3^2 + \cdots + \bar{l}_N^2 + 2\bar{l}_1\bar{l}_2 + 2\bar{l}_1\bar{l}_3 + \cdots$

+all possible cross terms

$< \bar{D}^2 > \ = < \bar{l}_1^2 > + < \bar{l}_2^2 > + < \bar{l}_3^2 > + \cdots + < \bar{l}_N^2 > + < 2\bar{l}_1\bar{l}_2 > + < 2\bar{l}_1\bar{l}_3 > + \cdots$

Cross terms average to 0

Others greater than 0

$= N l_{mfp}^2$

Average displacement root means squared

$\sqrt{< \bar{D}^2 >} = \sqrt{N}\, l_{mfp}$

$\sqrt{< \bar{D}^2 >} =$ RMS distance traveled after N steps in a random walk

Time taken for each step $= l_{mfp}/c$

Distance travelled after N steps $= \sqrt{N}\, l_{mfp}$

$=$Radius of sun (for photon to escape

$$= R_\odot \Rightarrow N = \frac{R_\odot^2}{l_{mfp}^2}$$

Total time for escape

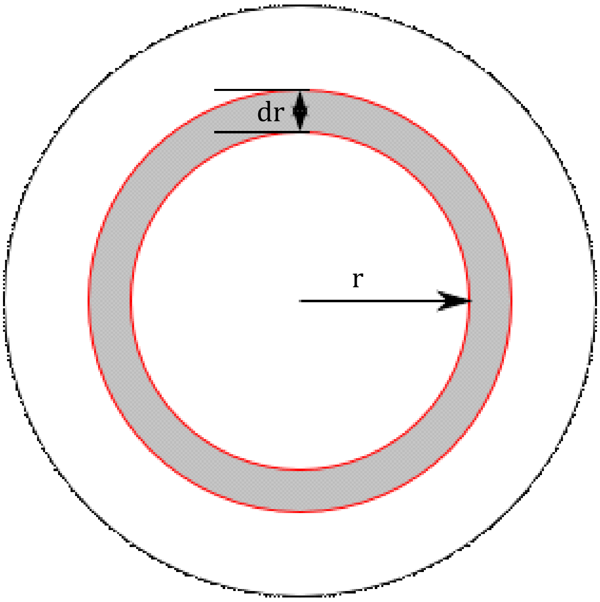
$$= N\frac{l_{mfp}}{c} = \frac{R_\odot^2}{l_{mfp}c}$$

$$l_{mfp} \approx 10^{-3}m$$

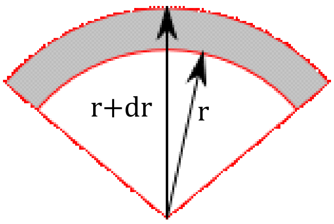
$$= 1.6 \times 10^{12}s \rightarrow 5 \times 10^4 years$$

Time taken for a photon to emerge from core of the sun to surface ~50,000 years

4. Radiative transfer



Outward flow of energy (neglect conduction & convection)



Energy density at "r" $= u(r)$

Energy density at "r+dr"$= u(r + dr)$

Total decrease of energy across the shell

$$= 4\pi r^2 dr\, du$$

Time taken by photons to cross the shell

$$= \frac{(\delta r)^2}{l_{mfp}c}$$

(random walk time)

∴ <u>rate</u> of energy transport

$$L(r) = -\frac{4\pi r^2 \delta u \delta r}{3\frac{(\delta r)^2}{l_{mfp}c}}$$

In the limit $\delta r \rightarrow 0$

$$L(r) = -\frac{1}{3}4\pi r^2 l_{mfp}c\,\frac{du}{dr}$$

Diffusion equation

$$l_{mfp} = \frac{1}{\kappa\rho}$$

$$\kappa = opacity$$

$\rho = density$

We know, for radiation

$u = aT^4$

$a = \text{radiation constant} = \frac{8\pi^5 k^4}{15c^3h^3}$

Plug into diffusion equation

$$L(r) = -\frac{l_{mfp}c}{3}\, 4\pi r^2 \times 4aT^3(r)\frac{dT}{dr}$$

$\frac{dT}{dr}$ =temperature gradient

$$\boxed{\frac{dT(r)}{dr} = -\frac{L(r)}{T^3(r)}\frac{3}{4\pi r^2}\frac{\kappa(r)\rho(r)}{4ac}}$$

Eqn of radiative transport

Use it to estimate $L_\odot$

$$\frac{T_\odot}{R_\odot} \sim \frac{L_\odot}{T_\odot^3}\frac{1}{R_\odot^2}\frac{1}{l_{mfp}ac}$$

$$\Rightarrow L_\odot \sim T_\odot^4 R_\odot l_{mfp} ac \sim 10^{26} W$$

$T_\odot^4 = estimate$

Compare to observed luminosity

$3.8 \times 10^{26} W$

Summary of 4 equations

1) $\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$
2) $\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$
3) $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$
4) $\frac{dT(r)}{dr} = \frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT^3(r)}$

Unknowns: $P(r), M(r), \rho(r), L(r), \epsilon(r), \kappa(r), T(r)$

Need 3 supplementary equations

Equation of state

$P = P(\rho, T, \text{chemical comp.})$

Stat mech

Opacity

$\kappa = \kappa(\rho, T, \text{chemical comp.})$

Scattering of light

Energy production

$\epsilon = \epsilon(\rho, T, \text{chemical comp.})$

Nuclear physics

Opacity

$$\frac{1}{\kappa\rho} = \frac{1}{n\sigma}$$

$$\Rightarrow \boxed{\kappa = \frac{r\sigma}{\rho}} = \frac{n\sigma_{thomson}}{\rho}$$ → Assuming scattering is dominated by free electrons

For stars with high T, all gas is ionized

$$\sigma_{thomspon} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2$$

n = number density of e^-

$$\kappa = \frac{n\sigma_{thomson}}{\rho} = \frac{\sigma_{thomson}}{m_H}$$

For stars entirely made from H

More generally

$$\kappa = \frac{\sigma_{thomson}}{m_H}\frac{1 + X_H}{2}$$

X_H=mass fraction of H in star

For low T stars

More complicated scattering S

Bound-bound scattering

Bound-Free scattering

Free-free scattering

$$\kappa \propto \frac{\rho}{T^{3.5}}$$

Scaling relations between L & M

05 March 2012
13:06

Opacity κ

- Constant for high T (thomson scattering)
- Low T $\sim\rho/t^{3.5}$ (Kramer's law)
- Low mass stars
  - $L \propto m^5$
- High mass
  - $L \propto m^3$

Explain this using eqn of radiative transport

Estimate for L

$$\sim \frac{T^4 Rac}{\kappa\rho}$$

From virial theorem

$$T \sim \frac{GM\overline{m}}{kR}$$

$\overline{m}$ = average mass of particle of the gass

$$\Rightarrow L \sim \frac{G^4 \overline{m}^4 M^4 Rac}{k^4 R^4 \kappa\rho}$$

For high T stars,

κ →constant

$$L \propto \frac{M^4}{\rho R^3} = M^3$$

Assuming T is high

Agrees with observations for high mass

Same analysis for low T using

$$\kappa \sim \frac{\rho}{T^{3.5}}$$

$$\Rightarrow L \propto \frac{M^{5.5}}{R^{0.5}}$$

Close to observations for low masses

Equations of state

$$P = P(\rho, T, composition)$$

Derive the most general equation of state

Take your fav gas

Suppose that number of particles/unit volume with momentum between $p\ \&p + dp = n(p)dp\ (all\ have\ momentum \sim |\bar{p}| = b)$

Total number of particles in the cylinder, moving to the right $= \frac{1}{6} n(p)dp$

In 1 sec, the momentum imparted to the wall (on right)

$$= \frac{1}{6} n(p)dp\ v(p) \times (2p) = \frac{1}{3} d(p) v(p) p\ dp$$

Pressure from particles with momentum between p and p+dp

Total pressure

$$\boxed{P = \int_0^\infty \frac{1}{3} d(p) v(p) p\ dp}$$

Works for all gases

<u>Check:</u>

Non relativistic ideal gas

$$v(p) = \frac{p}{m}$$

$$P = \frac{1}{3} \int_0^\infty dp\ n(p)\ p * \frac{p}{m}$$

[Ideal gas assumption]

$$= \frac{2}{3} \int_0^\infty dp\ \ n(p) * \frac{p^2}{2m} = \frac{2}{3} u$$

u=energy density

$$P = \frac{2}{3}u$$

$$PV = NkT \Rightarrow P = \frac{N}{V}kT = \left(n\frac{3}{2}kT\right)\frac{2}{3} = \frac{2}{3}u$$

Eqn of state for relativistic ideal gas (gas of photons)

$$v(p) = c$$

$$P = \frac{1}{3}\int_0^\infty dp\, n(p)pc$$

$$pc = E(p) = Energy$$

$$P = \frac{1}{3}\int_0^\infty dpn(p)E(p)$$

$$= \frac{1}{3}u$$

$$P = \frac{1}{3}u$$

Stability of stars and equations of state

Recall from virial theorem

$$\Omega = -3\int_{centre}^{surface} PdV$$

Ω = gravitational PE

For non-relativistic gas

$$\Omega = -3\int \frac{2}{3}udV$$

$$= -2\int udV$$

$$= -2\,U$$

$$\rightarrow \Omega + U = -2U + U < 0$$

For relativistic gas,

$$\Omega = -3\int \frac{1}{3}u\,dV = -U$$

$$\Rightarrow E_{total} = \Omega + U = 0!$$

A relativistic gas is not strictly bound by gravity

→ any "kick" to system will cause constituents to fly apart to infinity⇒ maximum possible mass for stars

→ find what point gas becomes relativistic & star stops being bound.

<u>Estimate</u>, Dimensional Analysis

$$P \sim \frac{GM^2}{R^4}$$

If gas becomes relativistic,

$$P = \frac{1}{3}$$

$$(U = aT^4)$$

$$aT^4 \leq \sim \frac{GM^2}{R^4}$$

We also know

$$kT \sim \frac{GMm_H}{R}$$

$$\rightarrow T = \frac{GMm_H}{kR}$$

$$\frac{aG^4M^4m_H^4}{k^4R^4} \leq \sim \frac{GM^2}{R^4} \Rightarrow M^2 \leq \sim \frac{k^4}{G^3m_H^4a}$$

$$M \leq \sim \frac{k^2}{G^{\frac{3}{2}}m_H\sqrt{a}} \approx 10^{31}kg$$

<u>More careful estimate</u>

$$P = P_{rad} + P_{kinetic} = \frac{aT^4}{3} + \frac{\rho}{\overline{m}}kT$$

P_{rad} due to radiation/photons

$P_{kinetic}$ ideal gas law

Star becomes "relativistic" when $P_{rad} = P_{kinetic}$

$$\frac{aT^4}{3} = \frac{\rho}{\bar{m}} kT$$

$$\rightarrow T = \left(\frac{3}{a}\right)^{\frac{1}{3}} \left(\frac{\rho k}{\bar{m}}\right)^{\frac{1}{3}}$$

We know that

$$\bar{P} \geq \frac{1}{8\pi} \frac{GM^2}{R^4}$$

Setting the 2 expressions for pressure equal to each other

$$2\left(\frac{3}{a}\right)^{\frac{1}{3}} \left(\frac{k}{\bar{m}} \frac{M}{\frac{4}{3}\pi R^3}\right)^{\frac{4}{3}} = \frac{GM^2}{8\pi R^4}$$

Solve for M

$$M = \frac{72\sqrt{3}}{\sqrt{4\pi}} \frac{1}{G^{\frac{3}{2}}} \frac{k^2}{\bar{m}^2 \sqrt{a}}$$

$$\approx 80 M_{\odot}$$

Equations of state

NR ideal gas

Relativistic ideal gas

Fermi + Bose gases

Fermions	Bosons
1/2 integer spin $\begin{pmatrix} Electron & \rightarrow & Spin \\ Proton/neutron & \rightarrow & \\ Quarks & \rightarrow & Spin \end{pmatrix}$	Integer spin $\begin{pmatrix} photon & \rightarrow & Spin\ 1 \\ Gluon & \rightarrow & \\ Higgs & \rightarrow & Spin\ 0 \end{pmatrix}$
Obey Pauli exclusion principle- No two fermions can occupy the same quantum state	More than one boson can occupy the same quantum state
Wavefunction is antisymmetric	Wavefunction is symmetric

<u>Fermi degeneracy pressure</u>

Pressure due to quantum mechanics

(due to Heisenberg uncertainty principle + Pauli Exclusion)

Fermions in a box.

Energy levels are quantized.

Fermions stack up on top of each other (in energy levels), even at very low temperatures ∴ they have large average KE (even at low T)

⇒ large momentum⇒ Fermi degeneracy pressure

$$P_{Fermi} = \frac{1}{3}\int_0^\infty dp\, n_F(p) * v(p) * p$$

$$n_F(p) = \frac{4\pi p^2}{h^3} \boxed{\frac{1}{e^{\frac{E-E_F}{kT}} + 1}} \times 2$$

E=energy

E_F =Fermi energy

$$\frac{1}{e^{\frac{E-E_F}{kT}} + 1}$$

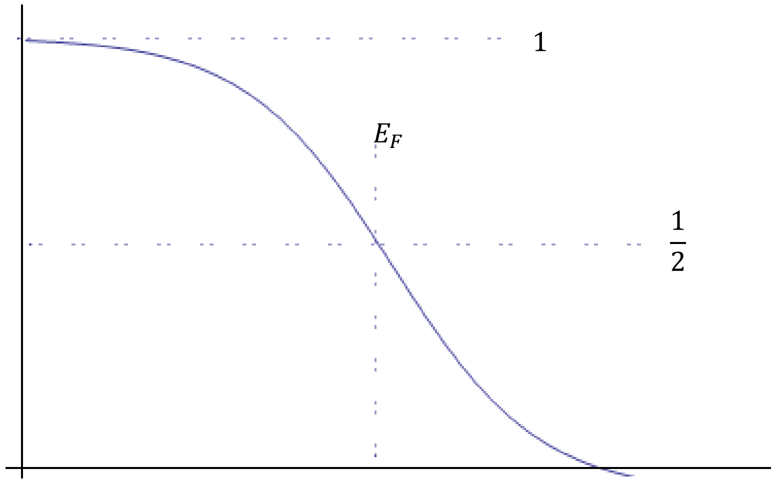
Plot

For any $T \neq 0$

At T=0, step function, goes to 0 at E_F

$$P = \frac{1}{3}\int_0^\infty dp \frac{8\pi p^2}{h^3} \frac{1}{e^{\frac{E-E_F}{kT}} + 1} p\, v(p)$$

$$f(E) \equiv \frac{1}{e^{\frac{E-E_F}{kT}} + 1}$$



$$P = \frac{1}{3}\int_0^\infty dp\ \frac{8\pi p^4}{h^3}\frac{1}{e^{\frac{\frac{p^2}{2m}+mc^2-E_F}{kT}}+1}$$

At low T, f€ becomes a step function

$$E_f - mc^2 \equiv \frac{p_f^2}{2m}$$

p_F = fermi momentum

~

Pressure of Fermi gases

$$P = \frac{1}{3}\int_0^{P_F}\frac{8\pi p^4}{h^3}dp$$

$$\boxed{P = \frac{1}{3}\frac{8\pi P_F^5}{5h^3}}$$

(1)

Number density of fermions

$$n = \int_0^{P_F}\frac{8\pi p^2}{h^3}dp$$

$$\boxed{n = \frac{8\pi P_F^3}{3h^3}}$$

(2)

From (1) and (2) $P \propto n^{\frac{5}{3}}$

For electrons in the star,

$$\boxed{P \propto \rho^{\frac{5}{3}}}$$

$$P_{fermi}^{NR} = \frac{h^2}{2om_e}\left(\frac{3}{\pi}\right)^{\frac{2}{5}}\frac{\rho^{\frac{5}{3}}(1+X_H)^{\frac{5}{3}}}{(2m_p)^{\frac{5}{3}}}$$

X_H = mass fraction of hydrogen

Independent of T

<u>Pressure of Fermi gases</u>

Relativistic (v=c, pc=E) fermi gas at low T,

$$P_{Fermi} = \frac{1}{3}\int_0^{P_F}\frac{8\pi p^2 dp}{h^3}pc$$

$$P_{Fermi} = \frac{1}{3}\frac{8\pi P_F^4}{4h^3}c$$

(1)

$$n = \frac{8\pi}{h^3}\frac{P_F^3}{3}$$

(2)

$$\Rightarrow P_{fermi} \propto n^{\frac{4}{3}}$$

$$\Rightarrow \boxed{P_{fermi} \propto \rho^{\frac{4}{3}}} \to \text{Relativistic Fermi Gas}$$

When does a fermi gas become degenerate?

Ans: When the typical interparticle separation is comparable to de Broglie wavelength of each fermion (electron)

$$\lambda = \frac{h}{p}$$

De Broglie wavelength

When

$$l \approx \frac{\lambda}{2}$$

Gas becomes degenerate

For an ideal NR gas, $\frac{p^2}{2m} \approx \frac{3}{2}kT \Rightarrow p \approx \sqrt{3mkT}$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{3mkT}}$$

Thermal wavelength

$n_e = \#$ density of electrons

$\therefore$ volume occupied by each electron $= \frac{2}{n_e}$

Typical distance between 2 electrons

$$\approx \left(\frac{1}{n_e}\right)^{\frac{1}{3}}$$

$$\Rightarrow l \sim n_e^{-\frac{1}{3}}$$

Matter in star becomes quantum/degenerate when

$$n_e^{-\frac{1}{3}} \approx \frac{h}{2\sqrt{3m_e kT}}$$

From this, we can obtain the critical density

$$n_e = \frac{8(3nkT)^{\frac{3}{2}}}{h^3}$$

$$\rho_{crit} = \frac{8m_p(3nkT)^{\frac{3}{2}}}{h^3}$$

What is ρ_{crit} in the sun's core?

$T_{core} \approx 1.5 \times 10^7 K$

$\rho_{crit} = 6.4 \times 10^5 kg/m^3$

Actual density of solar core

$\approx 1.5 \times 10^5 kg/m^3$

$\Rightarrow$ matter inside solar core is an ideal gas to a reasonable approximation

Main Sequence star

12 March 2012
13:45

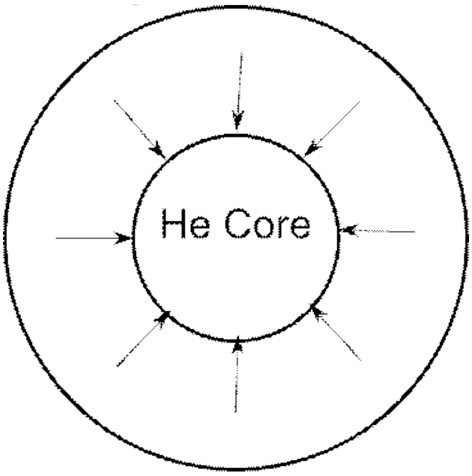
- Fuses $H\ to\ He$
  Most of the fusion takes place in the core region
  Virial theorem applies
  $\Omega = -2U$
  (Assumes gas is ideal)
  Core is helium rich (hydrogen turning into helium)
  As hydrogen fusion proceeds, He core becomes more massive
  → core undergoes contraction due to lack of hydrogen to fuse
  → more dense
  He core is prevented from complete collapse by Electron Degeneracy Pressure (EDP)

Compact objects supported by Fermi-Degeneracy pressure
White Dwarfs → Made of ^{12}C → stabilized by EDP (electron degeneracy pressure)
Neutron Stars→ Supported by Neutron DP

For objects supported by Fermi-Degenracy pressure
$P = k\rho^n$
n=4/3 or 5/3

Pressure for NR fermi gas (repeat for relativistic gas)
$P \propto \rho^{\frac{5}{3}}$
Each electron occupies a "cell" of width
$l \sim n_e^{-\frac{1}{3}}$
$\Delta x \Delta p \geq \hbar$
$\Delta p \sim \frac{\hbar}{n_e^{-\frac{1}{3}}} = \hbar n_e^{\frac{1}{3}}$
Kinetic energy
$\sim \frac{p^2}{2m_e} = \frac{\hbar^2 n_e^{\frac{2}{3}}}{2m_e}$
= typical KE for each electron
Pressure=
$\frac{1}{3}\int_0^\infty dP\ n(p)v(p)p$
$v(p) = \frac{p}{m_e}$
$= \frac{1}{3}\int_0^\infty dp\ n(p)\frac{p^2}{m_e}$
Pressure estimate
$\sim \left(\frac{\hbar^2}{m_e} n_e^{\frac{2}{3}}\right) \times n_e = \frac{\hbar^2}{m_e} n_e^{\frac{5}{3}} \propto \rho^{\frac{5}{3}}$

Core of main sequence star



Core gets smaller as H is "burnt"

Estimating when the core becomes degenerate

We know from Virial theorem

$$T_{core} \sim \frac{GM_{core}m_H}{kR_{core}}$$

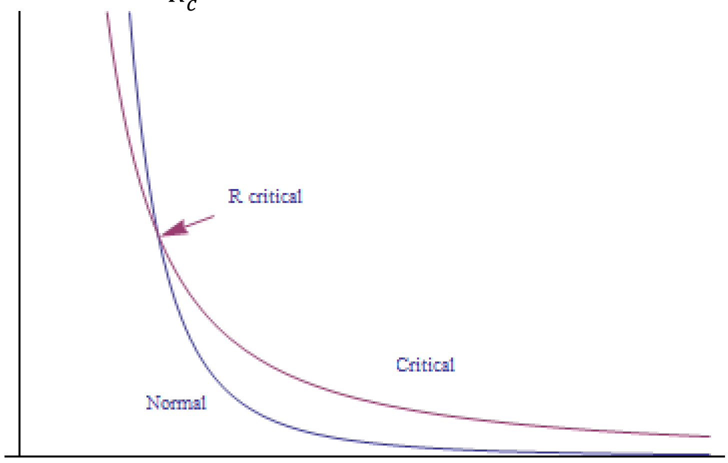
$$T_{core} \propto \frac{1}{R_{core}}$$

Core density $\propto \frac{M_c}{R_c^3}$

Critical density $\sim \frac{m_H(3m_e kT)^{\frac{3}{2}}}{h^3}$

→ beyond this core becomes degenerate

$$\rho_{crit} \propto T_c^{\frac{3}{2}} \propto \frac{1}{R_c^{\frac{3}{2}}}$$



Nuclear reactions

- Stars mostly made of H and He
    - Remnants of big bang
- Fusion is the only possible nuclear process
- Fusion releases energy because the heavier nuclei are tightly bound and their formation is exothermic.
- Binding energy (B): Energy supplied to break up a nucleus into constituent neutrons and protons
    - Neutrons are unstable (unlike protons). Bound neutrons don't decay because of the pauli exclusion principle. Unbound neutrons have a half life of ~15min

Nuclear Processes

19 March 2012
13:04

Binding energy

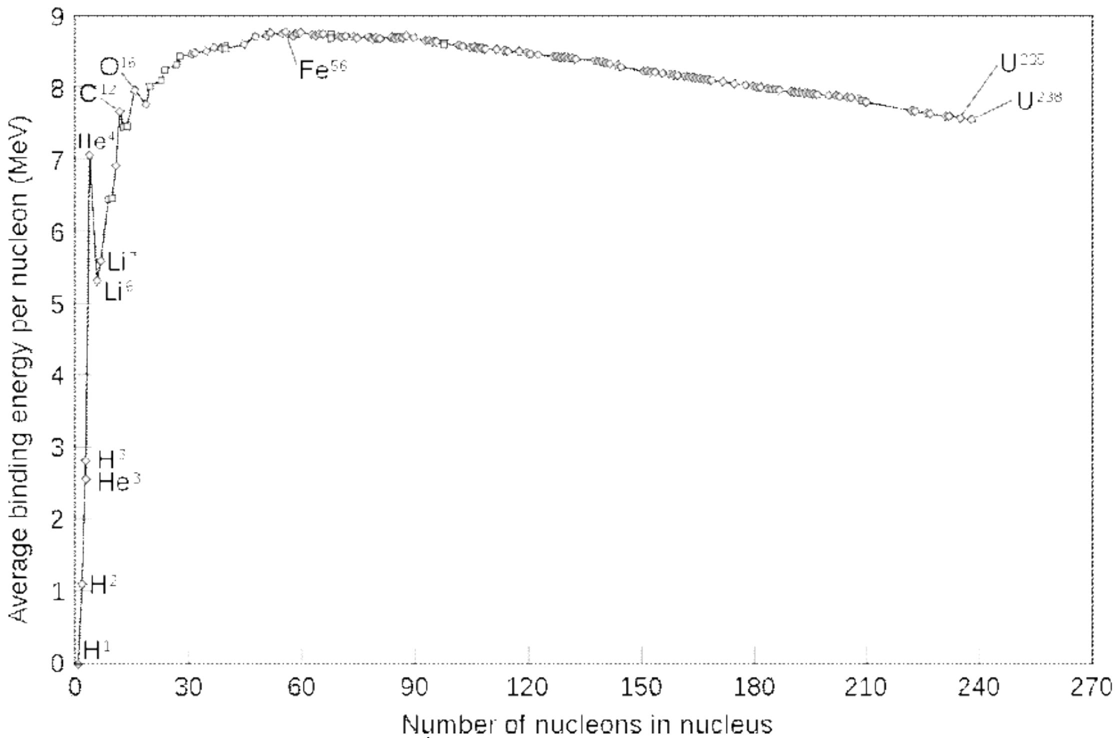
Given a nucleus with mass m_{nucl}, atomic mass number A (total # of protons +neutrons), and Z= number of protons.

Binding energy

$B = Zm_pc^2 + (A-Z)m_nc^2 - m_{nucl}c^2$

m_pc^2 rest mass energy of proton
m_nc^2 rest mass energy of neutron
$m_{nucl}c^2$ rest mass energy of nucleus

$B/_A$ = Binding energy per nucleon



Increase in $B/_A$ is the largest for $H \rightarrow He^4$

Low A nuclei would prefer to fuse to from heavy elements, progressively up to Fe^{56}

Nuclei with A>56, prefer to undergo fission

What happens inside a star?

Typical $T \sim 10^{6-7}k$

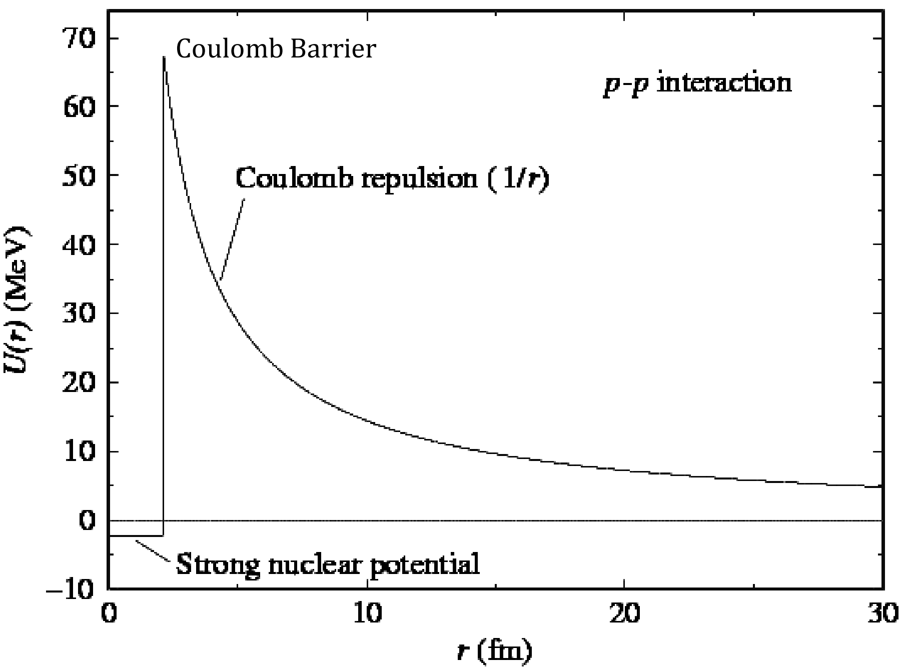
Typical energy $\sim kT = 1.38 \times 10^{-23}\frac{J}{k} \times 10^{6-7}k \sim 1000eV\ (keV)$

Two nuclei/protons need to overcome the electrostatic (coulomb) repulsion before they can fuse.

V(Inter-nuclear potential)

Coulomb barrier

$\frac{q_1q_2}{4\pi\epsilon_0 r_{min}}$



Coulomb barrier (for two protons)$\sim 1MeV$

→ problem: height of barrier is 1000x typical energy of protons in the star

Typical fraction of protons with energy E in the plasma at temperature T∝ $e^{-\frac{E}{kT}}$

Fraction of particles with $E\sim 1MeV$

$$= e^{-\frac{10^6 eV}{10^3 eV}} = e^{-1000} \sim 10^{-400}$$

Total no. of protons in sun $\sim 10^{57}$

Classically, fusion cannot occur in any star

→ Quantum mechanics responsible for fusion; proceeds by Quantum Tunnelling

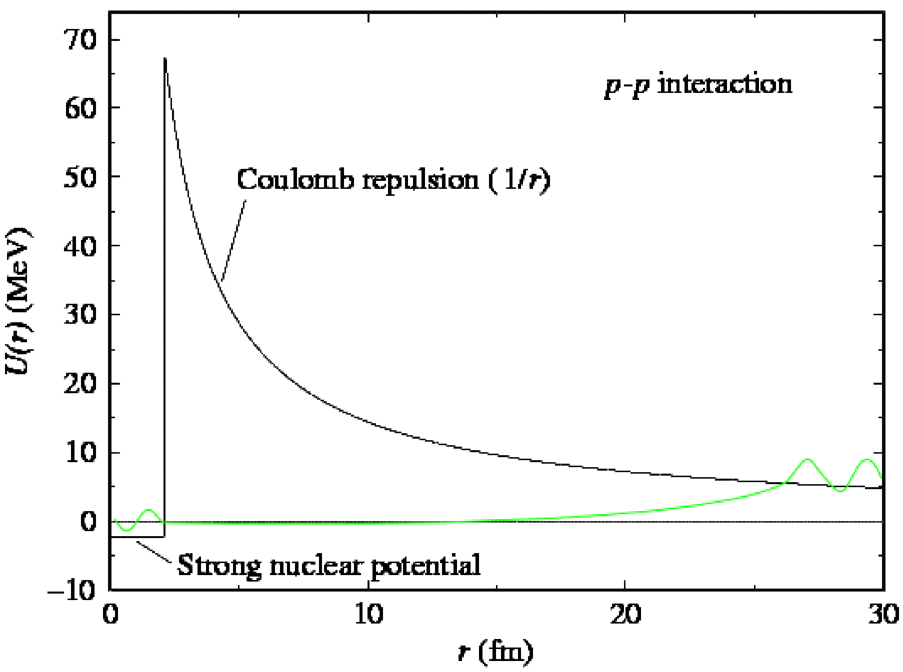
Quantum Tunnelling

- Given a potential V®
- Write schr eqn

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi$$

$$m_r = \frac{m_1 m_2}{m_1 + m_2}$$

- Solving for $\psi(r)$



- Probability for quantum tunnelling exponentially small

$$e^{-\sqrt{\frac{E_G}{E}}}$$

E_G=Gamow energy
E-energy of the nuclear pair undergoing fusion

$$E_G = \frac{2m_R}{\hbar^2}\ (e^2 Z_1 Z_2$$

$$X_1 + X_2 \rightarrow Y_1 + Y_2$$

Reaction state

$$= \int_0^\infty dV_r p(V_r)\sigma(V_r)V_r n_{X_1} n_{X_2}$$

dV_r – Relative velocity between nuclei X_1&X_2
$p(V_r)$ – Probability for tunnelling
$\sigma(V_r)$ – Interaction cross-section (Strong interaction physics), contains the Boltzmann distribution (ideal gas phase)
$n_{X_1} n_{X_2}$ – Number Densities

Tunnelling prob

$$= e^{-\sqrt{\frac{E_G}{E}}}$$

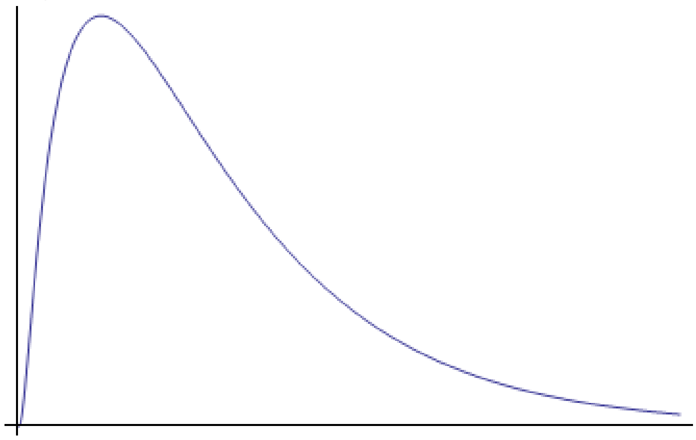
- Rapidly increasing function of E

Boltzmann factor

$$= e^{-\frac{E}{kT}}$$

- Rapidly decreasing function of E

$$e^{-\sqrt{\frac{E_G}{E}}} \times e^{-\frac{E}{kT}}$$



$$F(E) = e^{-\sqrt{\frac{E_G}{E}}} e^{-\frac{E}{kT}}$$

$$\frac{\delta F}{\delta E} = 0 \Rightarrow \boxed{E_{max} = E_G^{\frac{1}{3}}\left(\frac{kT}{2}\right)^{\frac{2}{3}}}$$

Saddle pt. approximation to integrals

$$\int_{-\infty}^{\infty} dx\ g(x) e^{-f(x)}$$

$e^{-f(x)}$ ←sharply peaked function
Say $f(x)$ has a maximum at $x = x_0$
Taylor expansion

$$f(x) \approx f(x_0) + (x - x_0)f''(x_0) + \frac{1}{2}(x - x_0)^2 \times f''(x_0)$$

$$e^{-f(x)} \approx e^{-f(x_0) - \frac{1}{2}(x-x_0)^2 f''(x_0) + \cdots}$$

$$= e^{-f(x_0)} \times e^{-\frac{1}{2}(x-x_0)^2 f''(x_0)}$$

$$f''(x_0) = Constant$$

$$\therefore e^{-\frac{1}{2}(x-x_0)^2 f''(x_0)} = gaussian$$

$$\int_{-\infty}^{\infty} dx\, g(x)e^{-f(x)} \approx_{\text{Saddle Pt. Approximation}} g(x_0)e^{-f(x_0)}\sqrt{\frac{2\pi}{f''(x_0)}}$$

$\sqrt{\frac{2\pi}{f''(x_0)}}$ – integrating Gaussian

$g(x_0)e^{-f(x_0)}$ – Evaluating the integrand at $x = x_0$

Reaction rate

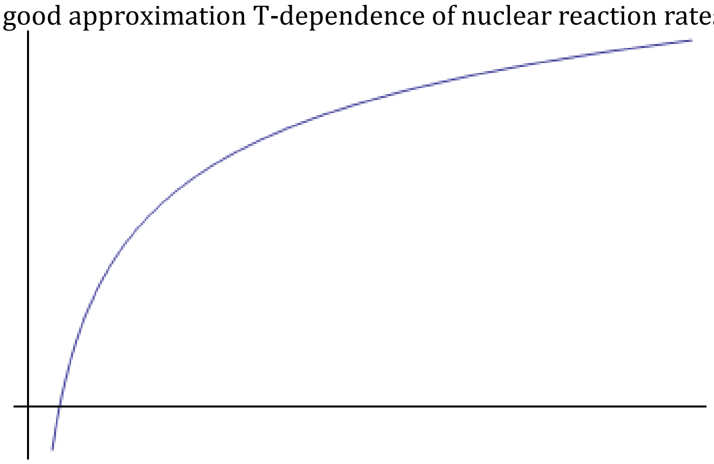
$$= e^{-\frac{E}{kT}}e^{-\sqrt{\frac{E_G}{E}}}\Big|_{At\ E=E_{max}} \times other\ stuff\ evaluated\ at\ E = E_{max}$$

$other\ stuff\ evaluated\ at\ E = E_{max} \leftarrow power\ law\ dependence\ on\ T$

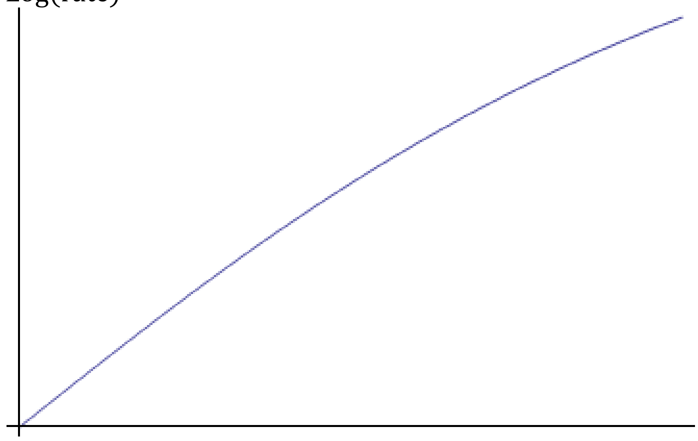
$e^{-\frac{E}{kT}}e^{-\sqrt{\frac{E_G}{E}}}\Big|_{At\ E=E_{max}} \leftarrow depends\ exponentially\ on\ T$

$$F(E = E_{max}) = e^{-3\left(\frac{E_g}{4kT}\right)^{\frac{1}{3}}}$$

To a good approximation T-dependence of nuclear reaction rates $\sim e^{-3\left(\frac{E_g}{4kT}\right)^{\frac{1}{3}}}$



Log(rate)



Slope of the curve obtained by plotting ln(rate) vs ln(T) (for any given temperature)

$$\nu = \frac{d(\ln(rate))}{d(\ln(t))} = \frac{T}{Rate}\frac{d(rate)}{dT} = \left(\frac{E_G}{4kT}\right)^{\frac{1}{3}}$$

Approximating the curve by a st. line

$\ln(rate) = \nu \ln T + constant$

Exponentiating both sides,

$\boxed{Rate \propto T^{\nu}}$

→ makes a power law approximation for T-dependence of rate

Main nuclear processes

i. p-p chain reaction (dominates in main sequence solar mass stars)
ii. C-N-O cycle (important in heavier stars)
iii. Helium burning →"triple α" process (Beyond main sequence stage)
iv. More advanced nuclear burning in very massive stars

p-p(I) chain

$p \quad + \quad p \quad \rightarrow \quad D^2 \quad + \quad e^+ \quad + \quad \nu_e$
$(H^1) \qquad (H^1)$

Any nuclear reaction must conserve charge, baryon number, lepton number

$D^2 + p \rightarrow {}^3He + \gamma$

γ=radiation/photons

$He^3 + He^3 \rightarrow He^4 + p + p$

Total energy released in this process per He^4 nucleus produced=26.7MeV→ a piece of this energy is due to e^+e^- annihilations $= 2 \times 2 \times 0.511MeV$

The rest of the energy released comes from binding energy of He^4

$4m_p c^2 - m_{^4He}c^2 = 25MeV's$ =0.7% of rest mass energy of 4 protons

$L_\odot \approx 3.8 \times 10^{26} W$

How many p-p fusion processes per second:

$$\frac{3.8 \times 10^{26} W}{4 \times 10^{-12} J} \approx 10^{38} processes$$

How many kgs of hydrogen /sec

$$10^{28} \times 10^{-27} = 10^{11} kg$$

$p + p \rightarrow D^2 + e^+ + \nu_e$

Governed by weak interactions, rate is comparatively low

Bottleneck processes

Primarily determines the reaction rates

$D^2 + p \rightarrow {}^3He + \gamma$

Strong interaction process

$He^3 + He^3 \rightarrow He^4 + p + p$

Strong interaction process

Reaction rate for p-p

$\epsilon_{p.p} \propto T^\nu$

$$\nu = \left(\frac{E_G}{4kT}\right)^{\frac{1}{3}} = \left(\frac{\frac{2m_r}{\hbar^2}\pi^2\left(\frac{e^2 Z_1 Z_2}{4\pi\epsilon_0}\right)^2}{4kT}\right)^{\frac{1}{3}}$$

$$m_r = \frac{m_1 m_2}{m_1 + m_2}$$

$$m_r = \frac{m_p^2}{2m_p} = \frac{m_p}{2}$$

$$Z_1 = Z_2 = 1, m_r = \frac{m_p}{2}$$

$$\nu = \left(\frac{0.49MeV}{4kT}\right)$$

$(E_G = 0.49MeV)$

$kT \approx 10^3 eV$

(roughly in the solar core)

$\Rightarrow \nu \approx 4.5 - 5$

Reaction rate for p-p

$\epsilon_{p.p} \propto T^{4.5} or\ T^5$

Nuclear rates are strongly sensitive to temperature

More massive stars: C-N-O cycle

Much hotter, so they contain a higher proportion of heavier elements such as C,N,O

H-fusion to He^4 is catalysed by C-N-O

$p + C^{12} \rightarrow N^{13} + \gamma$
$N^{13} \rightarrow c^{13} + e^+ + \nu_e$
$p + C^{13} \rightarrow N^{14} + \gamma$
$p + N^{14} \rightarrow O^{15} + \gamma$
$O^{15} \rightarrow N^{15} + e^+ + \nu_e$
$p + N^{15} \rightarrow C^{12} + He^4$

4 protons on left→ helium-4 on right

Same number of C, N, O on each side

Reaction rate is determined by $p + N^{14} \rightarrow O^{15} + \gamma$=bottleneck

$\epsilon_{CNO} \propto T^\nu = T^{18}$ at $T \approx 2.5 \times 10^7 k$

Typical core temperature of massive stars
Binding released $\approx 24.9MeV$
Similar to p-p chain
Reaction rates V. V. rapid due to T^{18} term

Stellar evolution stages

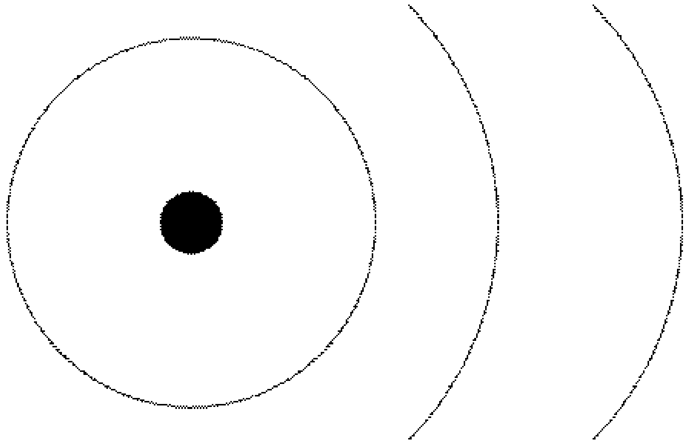
Solar mass stars

1) Proto-star (density exceeding Jeans density)
2) p-p chain reaction ignited in a core region
   - Entry into M-S stage, p-p reactions dominate in core
   - Core becomes He-rich as H gets used up
   - Core gradually contracts
     - By Virial theorem, it progressively gets hotter
   - Shells of H adjacent to core heat up
     - shells can undergo significant nuclear burning
   - "Shell burning" heats up outer layers, causing rapid expansion. (ideal gas expansion + outward pressure generated by photons released in shell burning)
   - Shell burning increases luminosity
     - T_{eff} decreases because of expansion
     - Star appears "redder"
       - Red giant stage
3) Red Giant phase
4) Helium burning
   - As R..G. progresses, He core contracts further until it is supported by EDP
   - Core contracts
     a. Heated by virial theorem
     b. Becomes a degenerate fermi gas

        $P = k\rho^{\frac{5}{3}}$

        (Eq of State)= T independent
   - Degenerate fermi gas has T-independent eq of state
     - Probability that there is a local region of the core where He-fusion occurs
     - →temperature increases locally
     - → This increase is transmitted to adjacent regions
     - →Spreads explosively
     - →Explosive spread of He^4 fusion to C^{12} known as He-Flash
     - $\epsilon_{He} \propto T^{40}$
   - Triple-α process
     1. $He^4 + He^4 \to Be^8$
     2. $Be^8 + He^4 \to C^{12*}$
     3. $C^{12*} \to C^{12} + \gamma \to radiation$

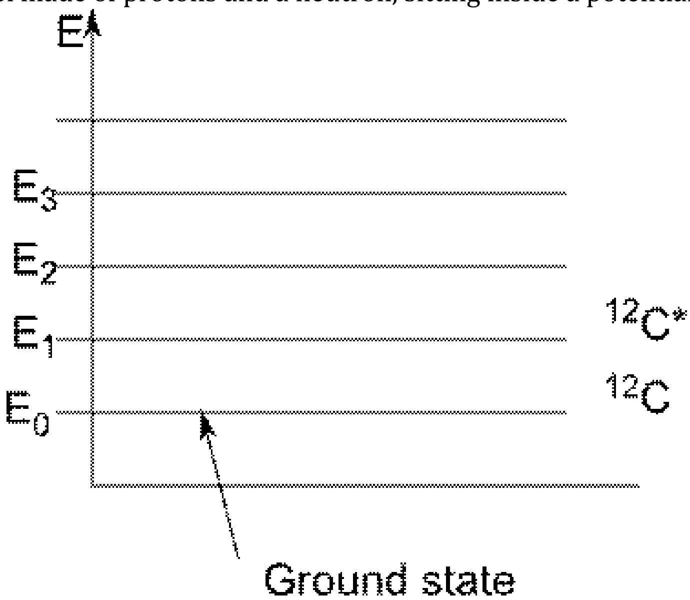
     $He^4 + He^4 \to Be^8$
     - Bottleneck
     - Rate calculation using Gamow energy doesn't work
     - Hoyle:
       - This method predicts too little C^{12} in observed universe, in comparison to what is actually measured
       - Prediction: Triple-α process is enhanced dramatically by a resonance
   - Resonance in QM:
     - H atom



Excitation of e^{-} occurs only if the difference of energy levels is hl

$\boxed{\Delta E = hl}$= enough energy to send it to next energy state

Nuclei made of protons and a neutron, sitting inside a potential well



Mass of B+He coincides almost exactly with excited state of C-12

-resonance

Rate is very large

5) Shell burning

+He-flash

Results in large amounts of stellar matter being ejected

He-burning: Core becomes C-rich

→Eventually, He is depleted from the core

→C-Core starts contracting

→Heats up

→Ignites adjacent shells

→Shell burning of He^4 to C^{12}

→Shells expand outward and are eventually ejected

Remnant: C^{12} core which contracts (no fusion reactions)

→becomes degenerate gas of e^{-}S

WHITE DWARF

There is a limiting mass for white dwarfs $\approx 1.5M_{\odot}$

Chandrasekhar limit

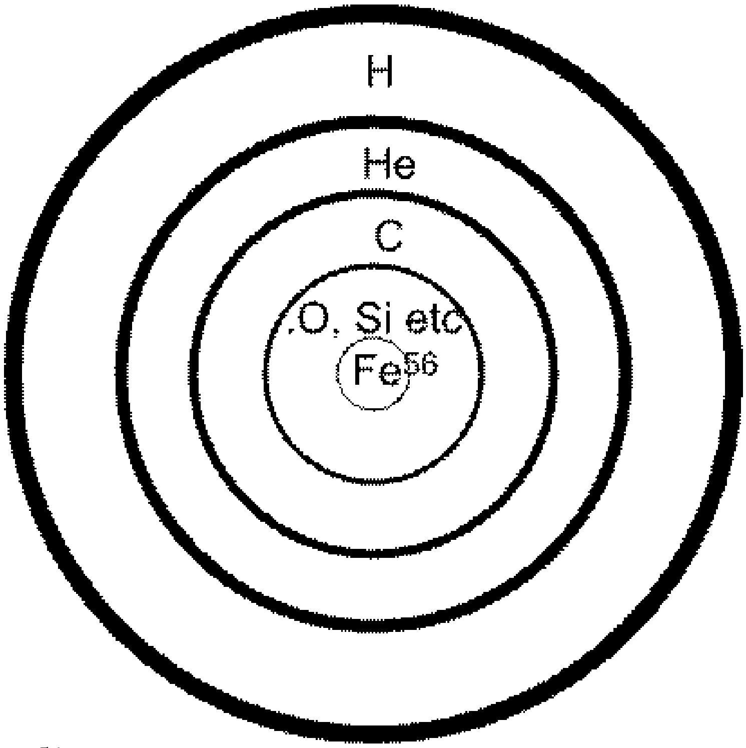
<u>Type 1a supernova</u>

Electron degeneracy pressure cannot support WD→collapsing

→Temperature increases→C^{12}-fusion

→C^{12} fusion occurs explosively-supernova

Very massive stars $(M > \sim 10M_{\odot})$



Fe^{56} core
→fusion stops in the core
→core contracts
→EDP is not sufficient to prevent collapse
→Further collapse(violently, large collapse), eventually pushing e^- s into nuclei→ $p + e^- \rightarrow n + \nu$
→all matter in core is converted to neutrons
→eventually, neutron degeneracy pressure stabilises the core
→matter from outer shells fall in and bounce off core→ shockwave→ type 2 supernova
Remnant: neutron star

White dwarfs and limiting masses

23 April 2012
13:05

White dwarfs- endpoint of evolution of solar mass stars
Described in term of degenerate electron matter
Supported by fermi pressure
WD's composed mainly of C^{12} (and O)→ matter is condensed and supported by EDP
Equation of state for degenerate electron gas feeds into hydro-equilibrium condition
Shown previously

$$P_{NR} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m_c}n_e^{\frac{5}{3}}$$

$$P_{relativistic} = \frac{1}{8}\left(\frac{3}{\pi}\right)^{\frac{1}{3}} hc\, n_e^{\frac{4}{3}}$$

Understand intuitively

$$P = \frac{1}{3}\int_0^{\infty} p(p)v(p)p\, dp$$ (Universal formula)

Scaling argument

$$\sim n_e \times \frac{p}{m_e} * p = En_e = \frac{p^2}{m_e}n_e$$

Thinking of electrons as particles in a box,

$$p \sim \frac{h}{\Delta x}\ (heisenberg\ uncertainty\ principle)$$

$$\sim hn_e^{\frac{1}{3}}$$

$$P \sim \frac{p^2}{m_e}n_e \sim \frac{h^2}{m_e}n_e^{\frac{5}{3}} \rightarrow NR\ case$$

Relativistic case

$$P \sim n_e p\, c \sim n_e c\, h\, n_e^{\frac{1}{3}} = hcn_e^{\frac{4}{3}}$$

Showed previously

$$P_{NR} = \frac{1}{20}\frac{\left(\frac{3}{\pi}\right)^{\frac{2}{3}} h^2}{m_e}n_e^{\frac{5}{3}}$$

Convert from number density n_e to mass density ρ

$$n_e = \frac{\rho X}{m_H} + \frac{\rho(1-x)}{2m_H}$$

$$x = mass\ fraction\ in\ Hydrogen\ originally$$

$$P_{NR} = \left(\frac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{20m_e}\frac{(1+X)^{\frac{5}{3}}}{(2m_e)^{\frac{5}{3}}}\rho^{\frac{5}{3}}$$

$$P_{Rel} = \left(\frac{3}{\pi}\right)^{\frac{1}{3}}\frac{hc}{8}\frac{(1+X)^{\frac{4}{3}}}{(2m_H)^{\frac{4}{3}}}\rho^{\frac{4}{3}}$$

$$P(r) = K\rho^n(r)$$

Equilibrium condition:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dM(r)}{dr}4\pi r^2\rho(r)$$

$$P(r) = k\rho^n(r)$$

Take derivative of 1st eqn

$$\frac{r^2}{\rho(r)}\frac{dP}{dr} = -GM(r)$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -\frac{GdM}{dr} = -G4\pi r^2\rho$$

$$\frac{d}{dr}\left(\frac{\frac{r^2}{\rho}dP}{dr}\right) = 4\pi G r^2 \rho(r)$$

Plugging in for P in terms of $\rho(r)$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}Km\rho^{n-1}\frac{d\rho}{dr}\right) = -4\pi G r^2\rho(r)$$

=2nd order differential equation for $\rho(r)$

$$\Rightarrow \frac{Kn}{(n-1)4\pi G}\Big|_{constant} * \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho^{n-1}}{dr}\right) = -\rho$$

Solution to this equation (numerical) describes a white dwarf

Assume there exists a "scalar solution"

$\rho = \rho_c f(r)$

ρ_c =constant determined by K, G, n

F(r)=dimensionless function

$f(r=0) = 1$

$f'(r=0) = 0$

Density is maximal at centre

Plug into previous eqn for $\rho(r)$

$$a^2\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df^{n-1}}{dr}\right) = -E$$

$$a^2 = constant = \frac{kn}{(n-1)4\pi G}\ \rho_c^{n-2}$$

Check that "a" has dimensions of length

A sets the radius of the white dwarf

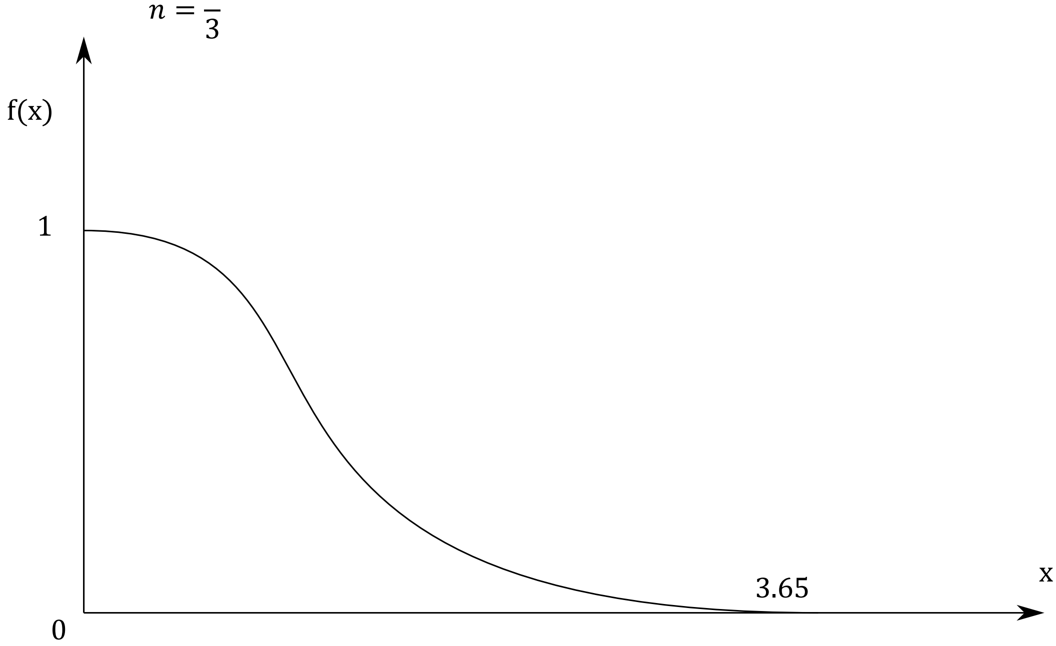
Useful to define

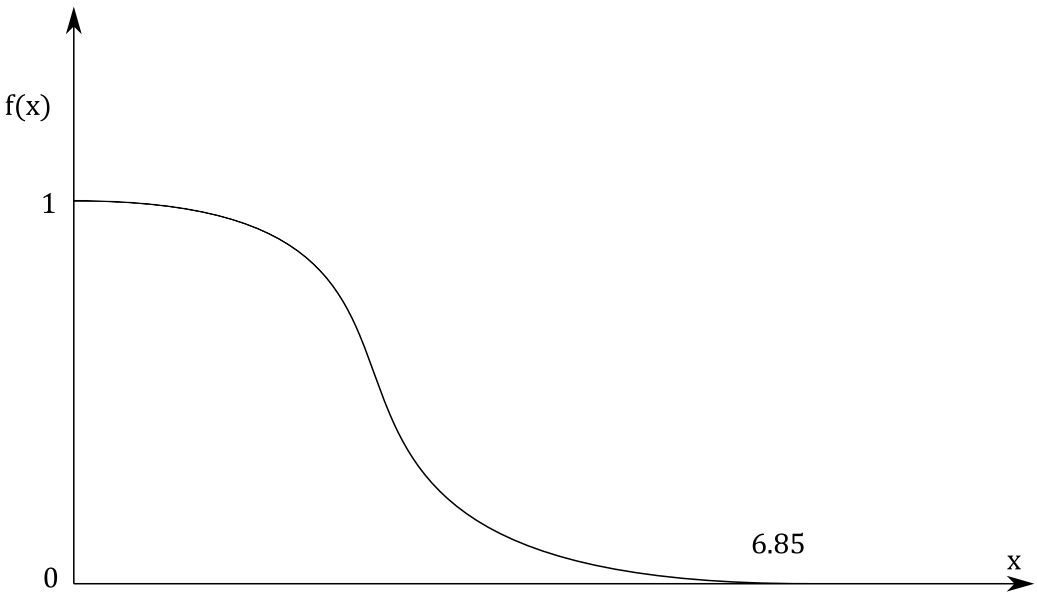
$$x \equiv \frac{r}{a}$$

$$\boxed{\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}f^{n-1}(x)\right) = -f(x)}$$

NR case

$$n = \frac{5}{3}$$





Mass of white dwarf

$$M_{WD} = \int_0^{R_{WD}} 4\pi r^2 \rho(r) dr = 4\pi a^3 \rho_c \int_0^{x_{max}} x^2 f(x) dx$$

For relativistic case

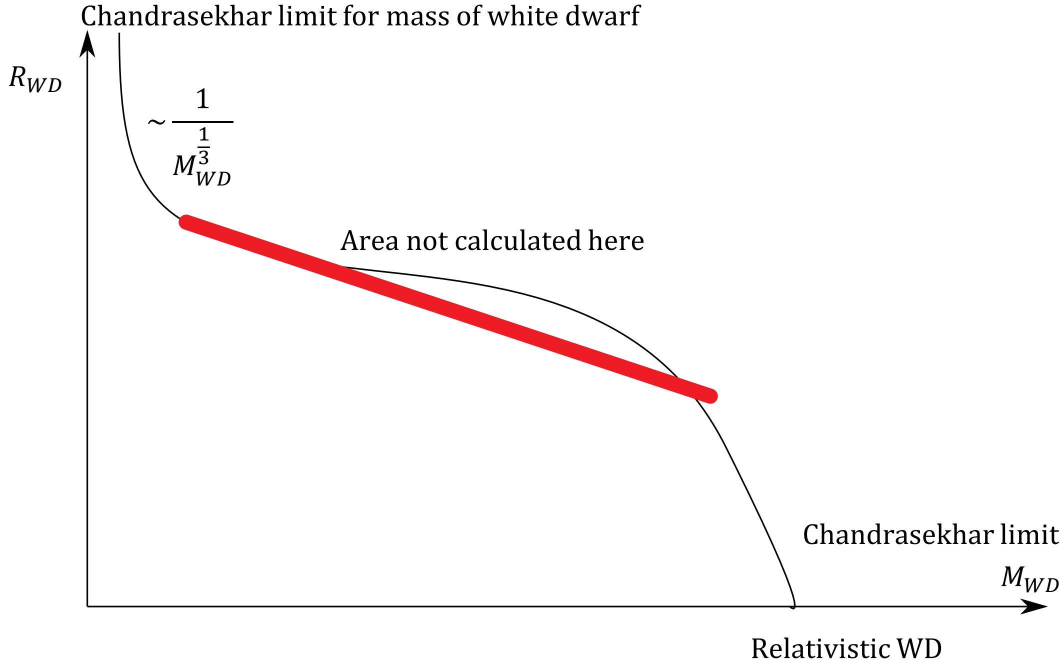
$$\int_0^{6.84} x^2 f(x) dx \cong 2.01$$

$$\Rightarrow M_{WD} = 4\pi a^3 \rho_c 2.01$$

Plugging in expressions for ρ_c and a

$$M_{WD} = \frac{\sqrt{3\pi}}{2} \left(\frac{hc}{2\pi G}\right)^{\frac{3}{3}} \frac{(1+x)^2}{4m_H^2} \times 2.01 \cong 1.43 M_{\odot}$$

Chandrasekhar limit for mass of white dwarf



Stars which are bigger than $1.5M_{\odot}$ will not be supported by EDP

Deriving the relation between M_{WD} and R_{WD} using dimensional arguments

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$P = k\rho^n$$

$$n = \frac{4}{3} \; or \; \frac{5}{3}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

Dimensional/scaling

$$\Rightarrow \frac{P}{R_{WD}} \sim \frac{GM_{WD}\rho}{R_{WD}^2}$$

$$\rho \sim \frac{M_{WD}}{R_{WD}^3}$$

$$\Rightarrow \frac{k\rho^n}{R_{WD}} \sim \frac{GM_{WD}\rho}{R_{WD}^2}$$

$$\Rightarrow \frac{kM_{WD}^n}{R_{WD}^{3n}} * \frac{1}{R_{WD}} \sim \frac{GM_{WD}}{R_{WD}^2} * \frac{M_{WD}}{R_{WD}^3}$$

$$\Rightarrow \boxed{M_{WD}^{n-2} \sim \left(\frac{G}{k}\right) R_{WD}^{3n-4}}$$

$$M_{WD}^{n-2} \sim \left(\frac{G}{k}\right) R_{WD}^{3N-4}$$

Non-relativistic: $n = 5/2$

$$M_{WD}^{-\frac{1}{3}} \sim \left(\frac{G}{k}\right) R_{WD} \Rightarrow R_{WD} \propto \frac{1}{M^{\frac{1}{3}}}$$

Relativistic: $n = 4/3$

$$M_{WD}^{-\frac{2}{3}} \sim \left(\frac{G}{k}\right) \Rightarrow M_{WD} = constant$$

EQ of state does NOT apply to neutron stars