Mathematical Methods II

30 January 2012
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80% exam
20% CE- 3 assignments
Course
Fourier series
Fourier Transform
Complex Analysis

=> Introduce mathematical material, which allows us to describe physical concepts & solve physical systems accurately
=> Emphasis on <u>applications</u>
Not focus on proofs

Fourier Series

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Step 1

Maths

Suppose f(x) is a periodic function

(x is a dimensionless real number)

With period 2π

$$f(x+2\pi)=f(x)$$

THEN (fourier's theorem)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$$

Where a_n, b_n are constants

(proof is non-trivial)

Proof by example

There are 3 ways to express this: (1)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n\cos(nx+\phi_n)$$

(2)

$$=\sum_{n=-\infty}^{\infty}c_n e^{inx}$$

(3)

SHOW forms are equivalent

- Need to remember

$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$
$$\cos(a+b)=\cos a\cos b-\sin a\sin b$$

So

$$A_n\cos(nx+\phi_n)=A_n\cos nx\cos\phi_n-A_n\sin nx\ \sin\phi_n$$

So (2) $\equiv$ (1) provided

$$a_n=A_N\cos\phi_n$$
$$b_n=-A_n\sin\phi_n$$

Take eqn(b)/eqn(a)

Express this

$$\frac{b_n}{a_n}=-\frac{\sin\phi_n}{\cos\phi_n}=-\tan\phi_n$$

$$\Rightarrow \boxed{\phi_n=-\tan^{-1}\frac{b_n}{a_n}}$$

So

$$\sum_{n=-\infty}^{\infty}c_n e^{inx}=c_0+\sum_{n=1}^{\infty}\left(c_n e^{inx}+c_{-n}e^{-inx}\right)$$

$$=c_0+\sum_{n=1}^{\infty}\left((c_n+c_n)\cos nx+i(c_n-c_{-n})\sin nx\right)$$

So comparing (1) and (2)

$$c_0=\frac{a_0}{2}$$

$$c_n+c_{-n}=a_n$$

=eq(c)

$$c_n-c_{-n}=b_n$$

=eq(d)

Note $eq(c)-ieq(b)=(c_n+c_{-n})+(c_n-c_{-n})=a_n-ib_n$

$\Rightarrow 2c_n = a)n - ib_n$

$$c_n = \frac{1}{2}(a_n - ib_n)$$

Note: Fourier series works for both real and complex values f(x)

$f(x): \mathbb{R} \to \mathbb{C}$

In form (1) a complex f(x) corresponds to a complex $a_n \& b_n$

In form (3), a real f(x) $\boxed{c_n = c_{-n}^*}$

Formulate for $a_n \& b_n$

We will need the identity

$$I_{n,m} = \int_0^{2\pi} \sin(nx)\sin(mx)\,dx$$

n,m integers

Integrate[Sin[n*x]*sin[m*x],{x,0,2Pi}]

We use

$$\sin a \sin b = \cos(a-b) - \cos(a+b)$$

$$I_{n,m} = \frac{1}{2}\int_0^{2\pi} \{\cos\left((\mathrm{n}-\mathrm{m})\mathrm{x}\right) - \cos\left((\mathrm{n}+\mathrm{m})\mathrm{x}\right)\}dx$$

$$= \frac{1}{2}\left[\frac{\sin\left((n-m)x\right)}{n-m}\right]_0^{2\pi} - \frac{1}{2}\left[\frac{\sin\left((n+m)x\right)}{n+m}\right]_0^{2\pi}$$

$$= \frac{1}{2}(0-0) - \frac{1}{2}(0-0) = 0$$

$$I_{n,m} = 0$$

$$if\ m+n \neq 0\ \&\ m-n \neq 0$$

$m - n = 0 \to n = m$

$$I_{n,m} = \int_0^{2\pi} \cos nx \cos nx\,dx$$

$$= \int_0^{2\pi} \cos^2 nx\,dx > 0$$

$$= \int_0^{2\pi} dx \frac{1}{2}(1 + \cos 2nx\,)$$

$$= \frac{1}{2}2\pi + \left[\frac{\sin 2nx}{2n}\right]_0^{2\pi}$$

For any periodic function

$f(x + 2\pi) = f(x)$

We can define $x \in [0, 2\pi)$

OR $x \in (-\pi, \pi]$

Note

$$\int_0^{2\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx$$

ALSO (exercise)

$$\int_{-\pi}^{\pi} \sin nx \sin mx = ?$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx = 0$$

Return to Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Take

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi} dx \frac{a_0}{2}\cos mx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx \cos mx + b_n \sin nx \cos mx\,dx\right)$$

$$= 0 + \sum_{n=1}^{\infty} \pi \delta_{n,m} a_n + 0$$
$$= \pi a_m$$

So

$$\boxed{a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx}$$

Exercise- check

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

a_0?

Take

$$\int_{-\pi}^{\pi} f(x) dx$$
$$= \frac{a_0}{2} \int_{-\pi}^{\pi} 1 \, dx + \sum_{n=1}^{\infty} \int (a_n \cos nx + b_n \sin nx) dx = \pi a_0$$
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Comments on even + odd functions

EVEN

$$f(-x) = f(x)$$

ODD

$$f(-x) = -f(x)$$

Explicitly

$$h(x) = \frac{1}{2}\big(f(x) + f(-x)\big)$$

Explicitly even

$$g(x) = \frac{1}{2}\big(f(x) - f(-x)\big)$$

Explicitly odd

$$h(x) - g(x) = f(x)$$

Multiplying together

$$f(x) = f_1(x) f_2(x)$$

X	Even	Odd
Even	Even	Odd
Odd	Odd	Even

IF $f(x)$ is ODD

$$\int_{-\pi}^{\pi} f(x) dx \; = 0$$

IF $f(x)$ is even

$$\int_{-\pi}^{\pi} f(x) dx = 2 \int_{0}^{\pi} f(x) dx$$

Fourier series of even & odd functions simplify

If f(x) is an even function

$$\Rightarrow b_n = 0$$
$$\& \; f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Proof

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \, f(x) \sin(nx)$$

Even * odd=odd

$=0$

Since integral is odd

If f(x) is odd

$a_n - a_0 = 0$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Proof

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x) \cos nx$$

Odd*even=odd

$= 0$

Since integral is odd

EXAMPLE

Periodic function with period 2π

$$f(x) = \begin{Bmatrix} 1 & x \in (0,\pi) \\ -1 & x \in (-\pi, 0) \end{Bmatrix}$$

"square wave"

So

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(xn)\, dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\cos n\pi}{n} - \left(-\frac{\cos 0}{n}\right)\right)$$

$$= \frac{2}{\pi n}[1 - \cos(n\pi)]$$

$\cos(n\pi) = (-1)^n$

$$b_n = \frac{2}{\pi}\frac{1}{n}[1 - (-1)^n]$$

$$b_1 = \frac{4}{\pi}$$

$$b_2 = 0$$

$$b_3 = \frac{4}{\pi}\frac{1}{3}$$

$$b_4 = 0$$

$$b_5 = \frac{4}{\pi}\frac{1}{5}$$

So

$$f(x) = \frac{4}{\pi}\left[\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right]$$

$$= \frac{4}{\pi}\sum_{r=1}^{\infty} \frac{1}{2n-1}\sin\big((2n-1)x\big)$$

Define a bit better what we mean by f(x)=

Define

$$f_p(x) = \frac{a_0}{2} + \sum_{n=1}^{p}(a_n \cos nx + b_n \sin nx)$$

Note that f_p is continuious

Then

$f_p(x) \to f(x)$

As $p \to \infty$

"almost everywhere"

"almost everywhere"

≡except at points where measure is zero

Note

$$\int_{-\pi}^{\pi}\left(f(x)-f_p(x)\right)dx \to 0$$

IF f(x) is continuous

$f_p(x) \to f(x)$ everywhere

Suppose function is discontinuous at point (a)

Suppose

$$\lim_{x\to a^-} \square = f^-(a)$$
$$\lim_{x\to a^+} \square = f^+(a)$$

If

$$f(x) = \sum_{n=1}^{\infty}\frac{a}{\pi}\left(\frac{1-(-1)^n}{n}\right)\sin(nx)$$

$$\sin A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}\sin\left((n+1)x\right)+\sin\left((n-1)x\right)dx$$
$$= \frac{1}{\pi}\int_0^{\pi}\sin\left((n+1)x\right)-\sin\left((1-n)x\right)dx$$
$$= \frac{1}{\pi}\left(\left[-\frac{\cos(n+1)x}{n+1}\right]_0^{\pi}-\left[-\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}\right)$$

$n \neq 1, -1$

$$= \frac{1}{\pi}\left(-\frac{[(-1)^{n+1}-1]}{n+1}-\left[-\frac{(-1)^{n-1}-1}{n-1}\right]\right)$$
$$= \frac{1-(-1)^{n+1}}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right]$$
$$= \frac{1-(-1)^{n+1}}{\pi}\left[\frac{(n-1)-(n+1)}{(n+1)(n-1)}\right]$$

n=1 we have to do separately

n=1

$$a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$
$$= 1/\pi\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$
$$= \frac{2}{\pi}\int_0^{\pi}\sin x\,dx$$
$$= \frac{2}{\pi}[-\cos(x)]_0^{\pi} = \frac{2}{\pi}[-\cos\pi+\cos 0]$$
$$= \frac{4}{\pi}$$

$|\sin x| = ?$

"rectified ac-signal"

E6-triangle wave

$f(x) = x$

$-\pi \le x \le \pi$

$f(x+2\pi) = f(x)$

$f(x)$ is odd $\Rightarrow a_n = 0, a_0 = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, x \sin nx$$
$$\int_{-\pi}^{\pi} x \sin nx = \left[-\frac{\cos nx}{n} x\right]_0^{\pi} - \int_{-\pi}^{\pi} -\frac{\cos nx}{n} 1$$
$$= \left[\frac{\pi}{n}\cos(n\pi) - \left(-\frac{-\pi}{n}\cos(-nx)\right)\right] + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx\, dx$$
$$b_n = \frac{\pi}{n}(-1)^n + \frac{1}{n}\left[\frac{\sin nx}{x}\right]_{-\pi}^{\pi}$$

Extension to other periods

Suppose

$f(x) = f(x - L)$

$f(t) = f(t + T)$

$x_{phy} \equiv x_{phy} + L$

Let

$$x_{maths} = 2\pi \frac{x_{phy}}{L}$$

x_{maths} is dimensionless &

$x_{maths} = x_{maths} + 2\pi$

$$f(x_{maths}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx_{maths}) + b_n \sin(nx_{maths})$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} x_{phy}\right) + b_n \sin\left(\frac{2\pi n}{L} x_{phy}\right)$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx_{maths} \cos(nx) + f(x_{maths})$$
$$a_n = \frac{2\pi}{\pi L}\int_{-\left(\frac{L}{2}\right)}^{\frac{L}{2}} dx_{phys} \cos\left(\frac{2\pi}{L}\, x_{phys}\right) f\left(x_{phys}\right)$$

Applications

Driven harmonic oscillator

Solution for sin/cos ⇒ general case

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$

m-mass

b-viscous cons

k=restoration term

F(t)=driving force

⇒ $F(t + T) = F(t)$

T=period

⇒two parts to problem

A. Solve when F(t)=0

"Solving homogeneous part"

If $x_1(t)$ *and* $x_2(t)$ are solutions, then so is $x_1 + x_2$

2nd order linear differential equation

⇒Two linearly independent solutions

$x(t) = a_1x_1(t) + a_2x_2(t)$

B. Find a particular solution for $F(t) \neq 0$

$x_p(t)$

General solution is $x(t) = x_p(t) + a_1x_1(t) + a_2x_2(t)$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + w_0^2 x = f(t)$$

(*)

$$\gamma \equiv \frac{b}{m}$$
$$w_0^2 \equiv \frac{h}{m}$$

$$\hat{F} \equiv \frac{F}{m}$$

$([\gamma] = [t]^{-1}, [w_0^2] = [t]^{-2})$

trick Use a complex x*

$x(t) = x_r(t) + ix_i(t)$

If $x(t)$ satisfies (*)

Try a solution

$$x(t) = Ae^{i\alpha t}$$

In equation, which gives

$(ix)^2 Ae^{i\alpha t} + \gamma i\alpha Ae^{i\alpha t} + \omega_0^2 Ae^{i\alpha t} = 0$

$[-\alpha^2 + i\alpha\gamma + \omega_0^2]Ae^{i\alpha t} = 0$

So

$\alpha^2 - i\alpha\gamma - \omega_0^2 = 0$

For a good solution

$$\left[ax^2 + bx + c \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$\left(\alpha - \frac{i\gamma}{2}\right)^2 - \left(\frac{i\gamma}{2}\right)^2 - \omega_0^2 = 0$$

$$\left(\alpha - \frac{i\gamma}{2}\right)^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

$$\alpha - \frac{i\gamma}{2} = \pm\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$$\alpha = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \frac{\gamma}{2} \pm \bar{\omega}$$

$$\omega_0^2 - \frac{\gamma^2}{4} \geq 0$$

$$x(t) = Ae^{i\alpha t} = Ae^{-\frac{\gamma t}{2}} e^{\pm i\bar{\omega}}$$

Real solution

$$e^{-\frac{\gamma t}{2}}[a_1 \sin\bar{\omega}t + a_2 \cos\bar{\omega}t] = x(t)$$

$$A\cos(\bar{\omega}t + \phi)\, e^{-\left(\frac{\gamma t}{2}\right)}$$

$$\omega_0^2 - \frac{\gamma^2}{4} < 0$$

$$\alpha = \frac{i\gamma}{2} \pm i\sqrt{\frac{\gamma^2}{4} - \omega_0^2} = i\alpha_\pm$$

$e^{i\alpha t} \Rightarrow e^{-\alpha_+ t}\ or\ e^{-\alpha_- t}$

Particular solution

For

$$F(t) = F_0\cos(\omega t) = Re|F_0 e^{i\omega t}$$

Try $x(t) = Ae^{i\omega t}$

Substitute in

$-\omega^2 Ae^{-i\omega t} + i\omega\gamma Ae^{i\omega t} + \omega_0^2 Ae^{i\omega t} = \hat{F}_0 e^{i\omega t}$

$\Rightarrow A[-\omega^2 + i\omega\gamma + \omega_0^2] = \hat{F}_0$

To satisfy this, A must be complex!

Note

$$-\omega^2 + i\omega\gamma + \omega_0^2 = re^{i\theta} = r\cos\theta + i\sin\theta\, r$$

$$\Rightarrow r^2 = (\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2 = r = \sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

$$\tan\theta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$$

So

$$A = A_0 e^{i\delta}$$

$$A_0 = \frac{\hat{F}_0}{r}$$
$$\delta = -\omega$$
$$A_0(\omega) = \frac{\hat{F}_0}{[(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}}$$
$$\tan\delta = \frac{\omega\gamma}{\omega^2 - \omega_0^2}$$
$$\omega^2 \ll \omega_0^2 \ \delta \sim 0$$
LET
$$x \equiv \omega/\omega_0$$
$$\omega = x\omega_0$$
$$A(x) = \frac{\hat{F}_0}{\omega_0^2\left[(x^2-1)^2 + \frac{\gamma^2}{\omega_0^2}x^2\right]^{\frac{1}{2}}}$$
Define
$$Q \equiv \frac{\omega_0}{\gamma}$$
$$A(x) = \frac{\hat{F}_0}{\omega_0^2\left[(x^2-1)^2 + \frac{x^2}{Q^2}\right]^{\frac{1}{2}}}$$
Q dimensionless !
$$A(x=1) = \frac{\hat{F}_0}{\omega_0^2\left[\frac{1}{Q^2}\right]^{\frac{1}{2}}} = \frac{\hat{F}_0 Q}{\omega_0^2}$$
Suppose we have a general periodic F(t)

e.g. square wave

e.g.
$$F(t) = \frac{F_0}{m}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right]$$
$$\Rightarrow x(t)$$
$$= A_1(\omega)\cos(\omega t + \delta_1) + \frac{1}{3}A_1(3\omega)\cos(3\omega t + \delta_3)$$
$$+ \frac{1}{5}A_1(5\omega)\cos(5\omega t + \delta_5)\ \ldots$$
Forced oscillation
$$F = F_0\cos(\omega t)\ x(t){:}\ A\cos(\omega t + \delta) + transients$$

$$A(x,Q) = \frac{1}{\left[(x^2-1)^2 + \frac{x^2}{Q^2}\right]^{\frac{1}{2}}}$$
$$x = \omega/\omega_0$$
$$Q = \frac{\omega_0}{\gamma}$$
$F(t)$=Square waves
$$F(t) = F_0\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t\right)$$
$$F(t) = F_0[A(\omega,Q)\sin\omega t + \frac{1}{3}A(3\omega,Q)\sin 3\omega t + \frac{1}{5}A(5\omega,Q)\sin 5\omega t$$
Response is NOT a square wave

For small x, amplitude of response has a resonance when $\omega = \omega_0$

But also if $3\omega = \omega_0$, we have a large resonance due to $A(3\omega,Q) = Q$

ω_0 get resonances: $\omega = \omega_0, \frac{\omega_0}{3}, \frac{\omega_0}{5}$

In general, get resonances $\omega = \frac{\omega_0}{n}, n = 1,2,3,\ldots$

With relative co-efficient b_n

At small x

$$A(x,Q) = \frac{1}{\left[(x^2-1)^2 + \frac{x^2}{Q^2}\right]^{\frac{1}{2}}} \approx 1$$

So response just resembles fourier series and response has same shape as force

$A(x,Q) \to 1\ as\ x \to 1$

Application: Kallza-clein theories

Recall

Particle in spacetime

$$\psi(x,t) = \exp\left[\frac{i(Et-px)}{\hbar}\right]$$

ψ =eigenstate of $i\hbar\frac{\delta}{\delta x} \equiv \hat{p}_x$

$$i\hbar\frac{\delta}{\delta x}\psi = P_x\psi$$

$$-i\hbar\frac{\delta}{\delta t}\psi = E\psi$$

Relativistically,

$$E^2 - |p|^2c^2 = m^2c^4$$

$$p_{\mu p}^{\mu} = m^2c^4$$

$$p_\mu p^\mu \equiv \left(p_0^2 - \sum_{i=x,y,z} p_i^2\right)$$

$$p_0 = \frac{E}{c}$$

Consider spacetime to be 5-dimensional

Idea Kallza-Klein

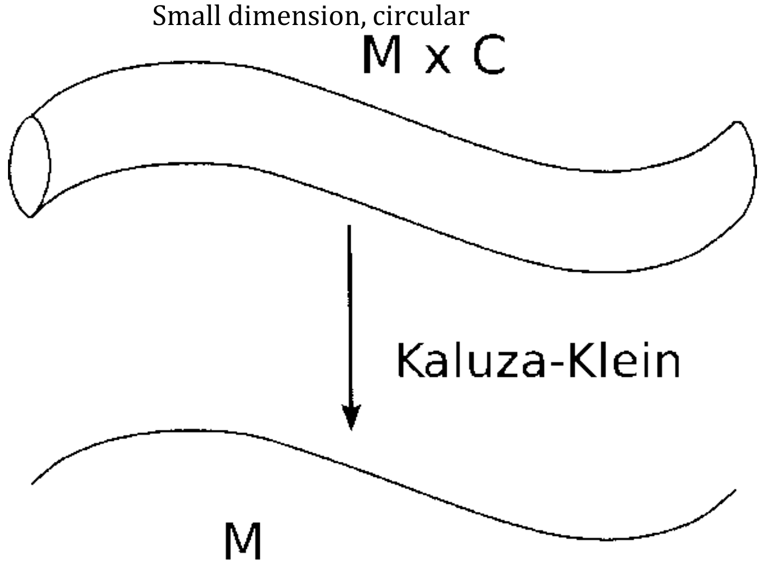
⇒1 time +4 spaces

-Extra dimension was condact

$0 \le x_5 \le 2\pi R$

(R small)

Small dimension, circular



From "a distance", space-time looks 4-dimensional

Consider a 5-D quantum particle

$$\frac{i}{\hbar}\left(Et - \underline{p}\underline{x} - p_5x_5\right)$$

$$\Psi = (\underline{x}, t, x_5) = e^{\frac{i}{\hbar}(Et-\underline{p}\underline{x}-p_5x_5)}$$

$$\hat{\mathrm{p}}_{\mathrm{x}}\Psi(\underline{x}, t, x_5) = \hat{p}_x\Psi$$

$$\boxed{it\frac{\delta}{\delta x_5}\psi = p_5\psi}$$

Since

$$0 \le x_5 \le 2\pi R$$

$$\psi(\underline{x}, t, x_5) = \sum_{n=-\infty}^{\infty} c_n(\underline{x}t)e^{\frac{i\hbar x_5}{R}}$$

Comparing to particle

$$\frac{P_5}{\hbar} \equiv \frac{n}{R}$$

Or

$$p_5 = \frac{\hbar}{R} \times n$$

=momentum in 5th d is quantized

=> in kallza klein electric charge recognised as 5th momentum

hgo

Momentum in 5th dimension is quantized in units of $\frac{\hbar}{R}$

$$\Rightarrow \hat{p}_5 e^{\frac{i\hbar x_5}{R}} = i\hbar \frac{\delta}{\delta x_5} e^{\frac{i\hbar x_5}{R}}$$

Gravity in 5-D

Gravity in 4-D +electromagnetism $[-Q \equiv R_5] = -\frac{\hbar n}{R}$

Note

$\sum P_N P^N = m^2$

n=1,2,3,4,5

$E^2 - p^2c^2 - p_5^2c^2 = m_0^2c^4$

p=3 momentum

$E^2 - |p|^2c^2 = m^2c^4 + p_5^2c^2 = m^2c^4 + \hbar^2n^2c^2/R^2$

=a 5-d particle looks like an infinite power of states in 4-D

Note

$\hbar = 6.6 \times 10^{-22} MeVs$

$c = 3 \times 10^8 m/\, s$

If

$r = 10^{-10}$

$$\frac{\hbar c}{R} = \frac{6.6 \times 10^{-22} MeVs \times 3 \times 10^8 m/\, s}{10^{-10}} = 1.96 \times 10^{-3} MeV$$

For $R = 10^{-15} m$

$$\frac{\hbar c}{R} = 1.96 \times 10^2 MeV$$

$\Rightarrow$ R<nuclear size

LHC experiment will search for masses up to ~TeV probably down to $10^{-19} m$

Kaluza-klein= very nice/beautiful idea, but no evidence observed in nature

Can "all" forces be viewed as gravity in higher dimensions??

Needs D=11 to work!

[11-D supergravity was very popular in 1980's]

K-K is an alternate way to acquire mass

If $m_0^2 = 0\; THEN\; m^2 > 0\; for\; n > 0$

Aside: New problem

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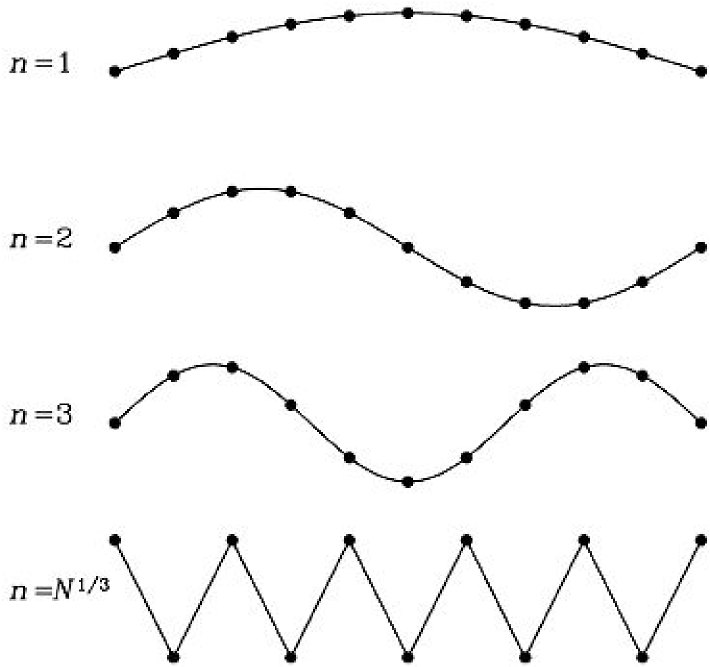
Phonons on a lattice



- Points $x = Na, N = -\infty, \ldots, a$
- Displacement defined $f(x) = Na$
- Consider a wave

  $f(x) = A\sin(kx - \omega t)$

  Note $R \equiv R'$

  $A\sin(kx - nt) = A\sin(k'x - \omega t)'$

  At $x = Na$

  $k'a = ka + 2\pi n$

  $k' = k + \frac{3\pi N}{a}$
- R only defined $-\frac{\pi}{a} < k < \frac{\pi}{a}$
- Known as brillouin zone
- $\omega(k)$ ≡dispersion relation

  Compare

k-k	Phonons
$0 \leq x \leq R$	$x = Na$
$p = \frac{\hbar n}{R}$	$R \in \left[\frac{0,2\pi}{a}\right]$

Harmonic analysis (not exactly fourier)

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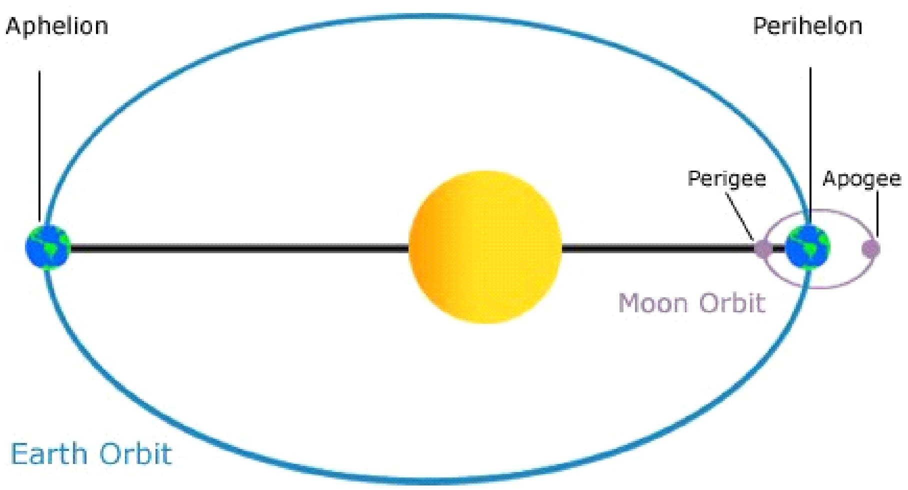
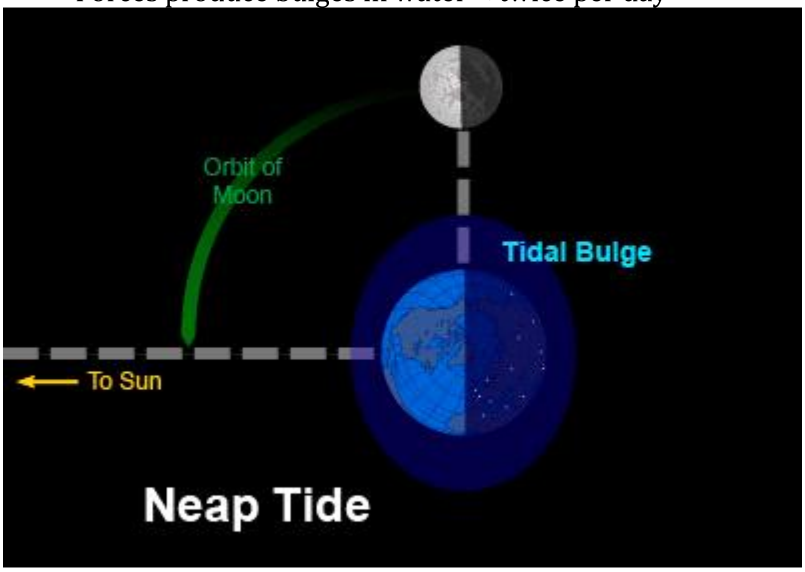
Example: tide
The physics of tides is quite complicated



- Forces produce bulges in water ⇒twice per day



- It is important that the system is moving. This would not work if the system is static

Look at this from the viewpoint of the water
- Upward force which is periodic
- $F(t) = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t)$

Model tides by

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Not a periodic problem unless

$$\frac{\omega_1}{\omega_2} = \frac{n}{m}$$

Force is not periodic

$F(t) = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t + \epsilon_2)$
We expect a response

$x(t) = A_1 \cos(\omega_1 t + \delta_1) + A_2 \cos(\omega_2 t + \delta_2)$

$A_1, A_2, \delta_1, \delta_2$

Tidal constants

Are extracted from data

$\omega_1 \& \omega_2$

$\omega = \frac{\pi}{T}$

For sun, T=0.5 days=12 hours

For moon, force has T=12.42 hrs

$W_{sun} = 0.523 hr^{-1}$

$W_{moon} = 0.5059 hr^{-1}$

$M_{sun} = 1.98 \times 10^{30} kg$

$M_{moon} = 7.3 \times 10^{22} kg$

~~

$$\frac{\text{Force of earth by sun}}{\text{Force of earth by moon}} = \frac{\frac{GM_eM_s}{R_S^2}}{\frac{GM_eM_m}{R_m^2}} = \frac{M_s}{M_m} \times \left(\frac{R_M}{R_s}\right)^2 = 2.5 \times 10^7 \times \frac{1}{400^2} = \frac{2.5 \times 10^7}{1.6 \times 10^8}$$

$= 1.5 \times 10^2$

Tidal forces

How much a fore varies across an object

Tidal force=$(F_1 - F_2)$

~~

$$\frac{\text{tidal force of earth by sun}}{\text{tidal force of earth by moon}} = \left(\frac{M_S}{M_m}\right)\left(\frac{R_m}{R_S}\right)^3 \sim \frac{150}{400} \sim \frac{1}{2.6}$$

Strongest tidal force due to moon (by a small factor)

What does this look like?

Special case: $A_1 = A_2$

$x(t) = A_1[\cos(\omega_1 t + \delta_1) + \cos(\omega_2 t + \delta_2)]$

$\cos A + \cos B$

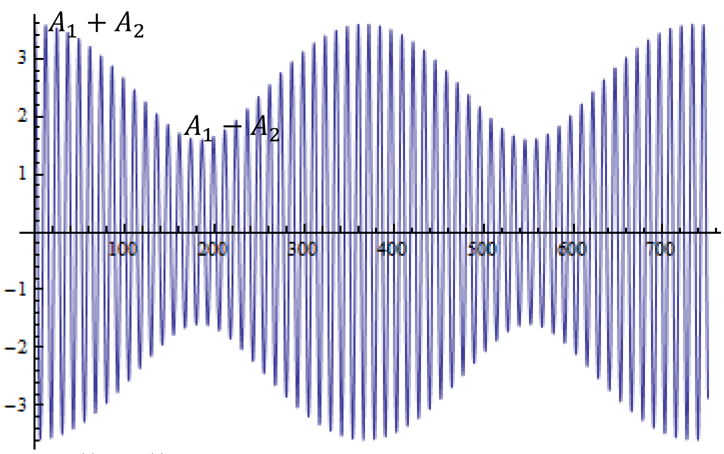
$$2A_1 \cos\left(\frac{(A+B)}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$= 2A_1 \cos\left(\left(\frac{\omega_1 + \omega_2}{2}\right)t + \left(\frac{\delta_1 + \delta_2}{2}\right)\right) \cos\left(\left(\frac{\omega_1 - \omega_2}{2}\right)t + \left(\frac{\delta_1 - \delta_2}{2}\right)\right)$$

Suppose $\omega_1 + \omega_2 \gg \omega_1 + \omega_2$

For us, $\omega_1 = 0.5059, \omega_2 = 0.523$

BUT $A_1 \neq A_2$ $(A_1 > A_2)$



$$\frac{\omega_1 - \omega_2}{2} = 0.008853 hr^{-1}$$

$$\Rightarrow T = \frac{2\pi}{\square} = 709.7\ hours = 29.57\ days$$

"envelope" occurs every 29.57 days

Small oscillations occur twice daily

Other effects

Elliptical orbits

$F(t) = F_0 \cos(\omega_1 t)\,[1 + \alpha \cos(\omega_3 t)]]$

T_3=365.25 days=4383hrs

$W_3 = 0.00143 hrs^{-1}$

α small

Effect of w3 oscillation is

$F(t) = F_0 \cos(\omega_1 t) + F_0 \alpha \cos\big((\omega_1 + \omega_3)t\big) + F_0 \alpha \cos\big((\omega_1 - \omega_3)t\big)$

⇒ response also has extra harmonic components

$x(t)_{extra} = A_3 \cos\big((\omega_1 - \omega_3)t + \delta_3\big) + A_4 \cos\big((\omega_1 + \omega_3)t + \delta\big)$

⇒ take these & other effects into account by adding more constants into response

⇒ tides can be predicted to $<1\%$

Fourier Transform

27 February 2012
12:11

- Use Fourier techniques for non-periodic functions
- Start with complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}$$

Minus sign important

- For period $2\pi L$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in}{L}x}$$

$$C_n = \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} f(x)e^{-\frac{in}{L}x}dx$$

- We want $L \to \infty$

Use

$$\hat{f}\left(\frac{n}{L}\right) = C_n$$

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{L}\right) e^{\frac{in}{L}x}$$

$$\hat{f}\left(\frac{n}{L}\right) = \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} f(x)e^{-\frac{in}{L}x}$$

Define

$$\tilde{f}\left(\frac{n}{L}\right) = \sqrt{2\pi}L\hat{f}\left(\frac{n}{L}\right)$$

$$\hat{f}\left(\frac{n}{L}\right) = \frac{1}{\sqrt{2\pi}}\frac{1}{L}\tilde{f}\left(\frac{n}{L}\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{L}\sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{n}{L}\right) e^{\frac{in}{L}x}$$

(*)

$$\tilde{f}\left(\frac{n}{L}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\pi L}^{\pi L} f(x)e^{-\frac{inx}{L}}$$

- Recognise (*) as discretization of integral

So

$$\int_{A}^{B} \tilde{f}(p)e^{ipx}$$

Discretised is

$\sum \tilde{f}(n\Delta x)e^{ip(n\Delta x)}$

With

$$\Delta x = \frac{1}{L}$$

So as $L \to \infty$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(p)e^{ipx}$$

(1)

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ipx}$$

(2)

2 is presented as a definition

1 is a result

Function $\tilde{f}(p)$ is the fourier transform of $f(x)$

$f(x) \mapsto \tilde{f}(p)$

Means $\tilde{f}(p)$ is the Fourier transform of f(x)

<u>Properties</u>

1. If $f(x) \mapsto \tilde{f}(p) | g(x) \mapsto \tilde{g}(p)$

   Then

   $\alpha f(x) + \beta g(x) \mapsto \alpha f(p) + \beta g(p)$

2. If $f(x) \mapsto \tilde{f}(p)$

   Then

   $\bar{f}^*(x) \mapsto \tilde{\bar{f}}^\star(p) = \tilde{\bar{f}}(-p)$

   $$\tilde{\bar{f}}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, \bar{f}(x) e^{-ipx}$$

   $$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} \tilde{f}(p)\right)$$

   $\tilde{\bar{f}}(-p)$

3. If $f(x) \mapsto \tilde{f}(p)$

   Then

   $f(x-a) \mapsto e^{-ipa}\tilde{f}(p)$

   Let $f(x) = f(x-a)$

   $$\tilde{g}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x-a) e^{-ipx}$$

   Let $y = (x-a); x = y + a$

   $dy = dx$

   $x = \pm\infty, y = \pm\infty$

   $$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy f(y) e^{-ipy} e^{-ipa}$$

   $$= e^{-ipa} \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, f(y) e^{-ipy} = e^{ipa}\tilde{f}(p)$$

4. If $f(x) \mapsto \tilde{f}(p)$

   $$f(\alpha x) \mapsto \frac{1}{|\alpha|}\tilde{f}\left(\frac{p}{\alpha}\right)$$

   Proof

   Let $g(x) = f(\alpha x)$

   $$\tilde{g}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx f(\alpha x) e^{-ipx}$$

   Let $y = \alpha x, dy = \alpha dx$

   $$3. \tilde{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{dy}{\alpha} f(y)\, e^{-i\frac{py}{\alpha}}$$

   If $\alpha > 0m\ x = \pm\infty, y = \pm\infty$

   $$= \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, f(y) e^{-i\left(\frac{p}{\alpha}\right)y}$$

   $$= \frac{1}{\alpha} f\left(\frac{p}{\alpha}\right)$$

   $$= \tilde{g}(p) = \frac{1}{\alpha}\, \tilde{f}\left(\frac{p}{\alpha}\right), \alpha > 0$$

   $\alpha < 0, x = \pm\infty \Rightarrow y = \mp\infty$

   $$\int_{-\infty}^{\infty} dx \to \int_{\infty}^{-\infty} \frac{dy}{\alpha} = -\frac{1}{\alpha}\int_{-\infty}^{\infty} dy$$

   $$\tilde{g}(p) = -\frac{1}{\alpha} f\left(\frac{p}{\alpha}\right), \alpha < 0$$

   So

   $$\tilde{g}(p) = \frac{1}{|\alpha|}\tilde{f}\left(\frac{p}{\alpha}\right)$$

Example

$f(x) = e^{-\alpha x^2}$

Gaussian

Width = $1/\sqrt{\alpha}$

Point α where $f(x) = e^{-1}$

Small α, wide spread

Large α, narrow

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} e^{ipx}$$

Step 1 Need

$$I(\alpha) = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} \left[\tilde{f}(0) = \frac{1}{\sqrt{2\pi}} I(\alpha)\right]$$

Find it by considering

$$I_2 = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} e^{-(x^2+y^2)}$$

→ 2d integrals

$Note: e^{-\alpha(x^2+y^2)} = e^{-\alpha x^2} e^{-\alpha y^2}$

$$I_2 = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} \int_{-\infty}^{\infty} dy\, e^{-\alpha y^2} = I^2(\alpha)$$

Recap

$f(x) = e^{-\alpha x^2}$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ipx} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ipx} e^{-\alpha x^2}$$

<u>Aside:</u>

$$I(\alpha) = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}$$

$$I_2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-\alpha(x^2+y^2)} = I^2(\alpha)$$

Changing to (r, θ) coordinates, $x^2 + y^2 = r^2$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \to \int_0^{\infty} dr \int_0^{2\pi} d\theta$$

$$I_2 = \int_0^{\infty} dr \int_0^{2\pi} d\theta\, r e^{-\alpha r^2} = 2\pi \int_0^{\infty} dr\, r e^{-\alpha r^2}$$

Let U= αr^2

$r = 0 \Rightarrow u = 0$

$r = \infty \Rightarrow u = \infty$

$du = 2\alpha r dr$

$$= 2\pi \int_0^{\infty} \frac{du}{2\alpha} e^{-u}$$

$$\frac{\pi}{\alpha} [-e^{-u}]_0^{\infty} = \frac{\pi}{\alpha}$$

$$\Rightarrow I(\alpha) = \sqrt{\frac{\pi}{\alpha}}$$

Now

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} e^{-ipx}$$

We "complete the square" of $-\alpha^2 - ipx$

$$= -\alpha\left(x + \frac{ip}{2\alpha}\right)^2 + \alpha\left(\frac{iP}{2\alpha}\right)^2 = -\alpha\left(x + \frac{ip}{2\alpha}\right)^2 - \frac{p^2}{4\alpha}$$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} e^{-\frac{p^2}{4\alpha}} \int_{-\infty}^{\infty} dx\, e^{-\alpha\left(x + \frac{ip}{2\alpha}\right)^2}$$

Let $y = x + \frac{ip}{2\alpha a}$

$dy = dx$

$x = \pm\infty \Rightarrow y = \pm\infty$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{p^2}{4\alpha}} \int_{-\infty}^{\infty} dy e^{-\alpha y^2} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{\alpha}} e^{-\frac{p^2}{4\alpha}} = \frac{1}{\sqrt{2\alpha}} e^{-\frac{p^2}{4\alpha}}$$

Fourier transform of Gausian, width $= \frac{1}{\sqrt{\alpha}}$, is a Gausian with width $2\sqrt{\alpha}$

Example

$$f(x) = \begin{cases} 1 & |x| \le a \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx f(x) e^{-ipx}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} dx\, e^{-ipx} = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-ipx}}{-ip}\right]_{x=-a}^{x=a}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{-ip}\left[e^{-ipa} - e^{ipa}\right] = \frac{1}{\sqrt{2\pi}}\frac{2\sin pa}{p} = \frac{2a}{\sqrt{2\pi}}\frac{\sin(pa)}{pa} = \frac{2a}{\sqrt{2\pi}}\operatorname{sinc} pa$$

$\frac{\sin x}{x}$ $at\ x = 0\ is?$

We can use L'Hopitals theorem

$$\frac{f(x)}{g(x)} \to^{x\to 0} \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

In this case

$$\lim_{x\to 0}\frac{\cos x}{1} = 1$$

Both f(0) and g(0)=0

Example

$$f(x) = \begin{cases} |x-d| \le a & f(x) = 1 \\ |x+d| \le a & f(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note

$$f(x) = f_0(x-d) + f_0(x+d)$$

f_0 was example previously looked at

So

$$\tilde{f}(p) = \tilde{f}_0(p)e^{ipd} + \tilde{f}_0(p)e^{-ipd}$$

$$= \tilde{f}_0(p) \times 2\cos(pd)$$

$$= \boxed{\frac{4}{\sqrt{\pi}}\frac{\sin p\alpha}{\alpha}\cos pd}$$

For $d \gg \alpha\ \hat{f}(p)$

Any function can be approximated by step functions

Application

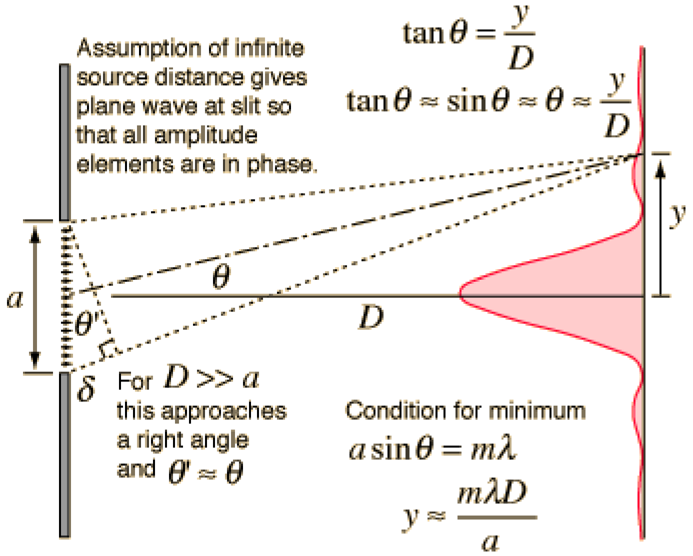
=Fraunhofer diffraction

Diffraction Fraunhoffer ≡ light & screen are effectively at ∞

Fresnel ≡ they are not

Consider diffraction through a slit (specialisation of 2-d problem)

To calculate light at Ym we imagine every point a is a source of light & then we combine resultant



Light is a wave that oscillates

$\sim \sin(\omega t - kx)$

$\cos\left(\omega t - \frac{2\pi}{\lambda}x_t\right)$

$= Re\left[e^{-\left(\omega t - \frac{2\pi}{\lambda}x_t\right)}\right]$

Distance of travel $x_T = r - d = r - x\theta$

So adding waves

$\int_{-a}^{a} dx\, e^{i\left(\omega t - \frac{2\pi}{\lambda}x_t\right)} e^{\frac{i2\pi x\theta}{\lambda}}$

$= e^{i\left(\omega t - \frac{2\pi}{\lambda}\theta\right)} \times \int_{-a}^{a} dx\, e^{\frac{i2\pi x\theta}{\lambda}}$

f(x)=source

A(θ)=image on screen

$A(\theta) = \int_{-\infty}^{\infty} dx\, f(x) e^{-i2\pi\frac{\theta}{\lambda}x}$

$= \tilde{f}\left(2\pi\frac{\theta}{\lambda}\right)$

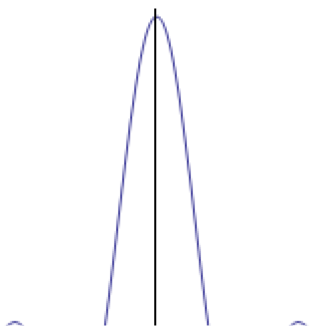
$f(x) \equiv transmission =$

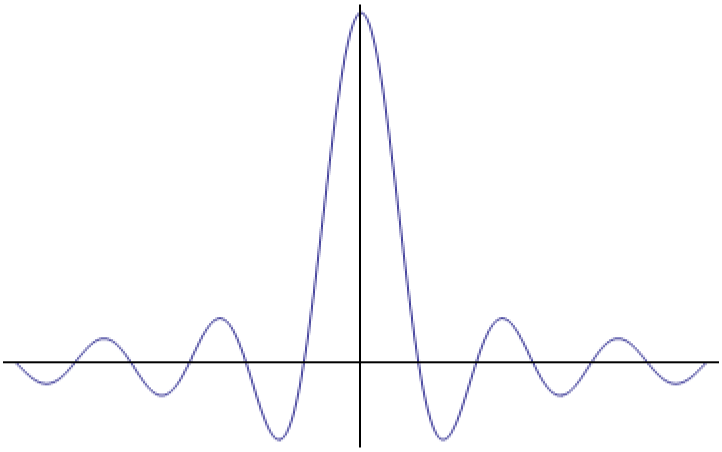
e.g. f(x)= square wave from -a to a

We get a pattern

$\tilde{f}\left(2\pi\frac{\theta}{\lambda}\right)$

$\frac{\sin(ka)}{k}$

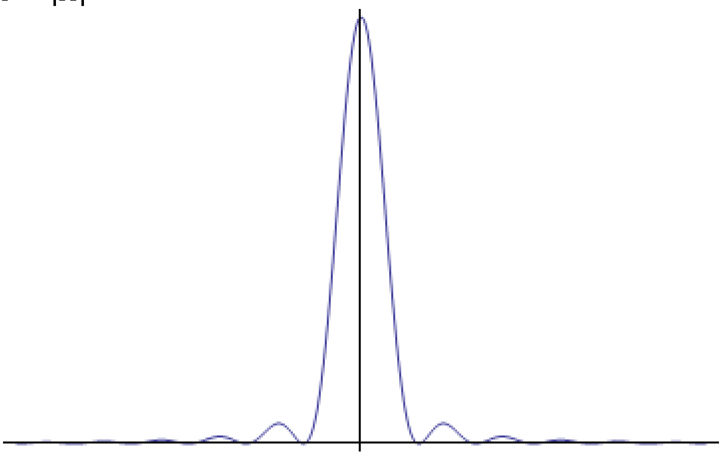




$\theta_{\text{first minimum}} = \frac{\lambda}{2a}$

Intensity of light

$I \propto |A|^2$



Q: how large is secondary maximum?

A. Find by taking derivative

$$\left[\frac{\sin x}{x}\right]'$$

$$\Rightarrow \frac{\cos x}{x} + \sin x - \frac{1}{x^2} = 0$$

$$\Rightarrow \frac{\cos x}{x} = \frac{\sin x}{x^2}$$

$$\Rightarrow x = \tan x$$

Solve numerically

2nd maximum has height, (intensity)=0.047

Double slit= single slit + single slit with phase shift

Solution

$$\left(e^{ikd} + e^{-ikd}\right) \times \frac{\sin ka}{k}$$

$$= \frac{2\cos kd \times \sin ka}{k}$$

$$k \Rightarrow 2\pi \frac{\theta}{\lambda}$$

Q: how large is secondary maximum

$f_{\text{Screen Image}} \rightarrow \tilde{f}_{\text{Intensity pattern}}$

$f_{\text{Screen Image}} \leftarrow \tilde{f}_{\text{Intensity pattern}}$

Can get from one to the other

Multi Dimensional Fourier Transform

12 March 2012
10:39

$$\tilde{f}(p_1, p_2) \equiv \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; e^{i(p_1 x + p_2 y)} f(x, y)$$

Application ∴ diffraction

$$A(\theta, \phi) = \tilde{f}\left(\frac{2\pi\theta}{\lambda}, \frac{2\pi\phi}{\lambda}\right)$$

Example

$$f(x, y) = \begin{Bmatrix} 0 & & \\ 1 & if & |x| < a \\ & And & |y| < b \end{Bmatrix}$$

$$\tilde{f}(p_1, p_2) = \tilde{f}_a(p_1) * \tilde{f}_b(p_2) = \frac{\sin p_1 a}{p_1} * \frac{\sin p_2 b}{p_2}$$

So

$$A(\theta, \phi) \sim \frac{\sin\left(\frac{2\pi\theta}{\lambda} a\right)}{\frac{2\pi a}{\lambda}} \frac{\sin\left(\frac{2\pi\phi}{\lambda} b\right)}{\frac{2\pi b}{\lambda}}$$

Properties of Fourier Transform

Suppose

$$f(x) \mapsto \tilde{f}(p)$$

$$xf(x) \mapsto i\frac{d}{dp}\tilde{f}(p)$$

$$\frac{df(x)}{dx} \mapsto ip\,\tilde{f}(p)$$

Proof

Let $g(x) = xf(x)$

$$\tilde{g}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\; e^{-ipx} x f(x)$$

$$\frac{d}{dp}\left[e^{-ipx}\right]$$

$$\left[\frac{1}{-i} = i\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\; i\frac{d}{dp} e^{-ipx}\; f(x)$$

$$= i\frac{d}{dp}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\; e^{-ipx} f(x)\right]$$

$$= i\frac{d}{dp}\tilde{f}(p)$$

Next

$$\left(\frac{df}{dx} \to ip\tilde{f}(p)\right)$$

Proof

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\; e^{ipx}\tilde{f}(p)$$

$$\frac{df}{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, \frac{d}{dx}[e^{ipx}\tilde{f}(p)$$

So

$$\frac{df}{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\; ip\; e^{ipx}\tilde{f}(p)$$

So

$$\frac{df}{dx} \mapsto ip\tilde{f}(p)$$

E.g.

$$e^{-\frac{x^2}{2}} \mapsto e^{-\frac{p^2}{2}}$$

$$xe^{-\frac{x^2}{2}} \mapsto \frac{d}{dp} e^{-\frac{p^2}{2}} = -ipe^{-\frac{p^2}{2}}$$

$$x^2 e^{-\frac{x^2}{2}}$$

$$= x * xe^{-\frac{x^2}{2}} \mapsto i\ dp\ \left[-ipe^{-\frac{p^2}{2}}\right] = (-p^2 + 1)e^{-\frac{p^2}{2}}$$

$$x^N e^{-\frac{x^2}{2}} \mapsto H_N(p)e^{\frac{-p^2}{2}}$$

$H_N(p)$= hermite polynomials

Validity of Fourier transform

$\tilde{f}(p)$ is a well defined function if

$$\int_{-\infty}^{\infty} dx |f(x)|^2 < \infty$$

(*)

Note

$$f(x) \to 0\ as\ x \to \pm\infty$$

Dirac δ- function

13 March 2012
14:03

{Dirac Delta}

We want to use Fourier transform when (*) (last page) doesn't hold
Not a function
$\delta(x)$ is a <u>measure</u> (or distribution)
A measure has well defined integrals
If μ is a measure
$\int dx\, \mu(x) f(x) =$ well defined
$\delta(x)$ Basic defining property

$$\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0)$$

$\delta(x)$ is a limit of normal functions

e.g. $\delta = \begin{Bmatrix} \frac{1}{2a} & -a < x < a \\ 0 & otherwise \end{Bmatrix}$

$\delta_a(x) \to \delta(x) as\ a \to 0$

NB

$$\delta(x) = \begin{Bmatrix} \infty & x = 0 \\ 0 & otherwise \end{Bmatrix}$$

Poorly defined

Consider

$$I_a = \int_{-\infty}^{\infty} dx\, \delta_a(x) f(x)$$

Look at integral

$$I_a = 2a \times \frac{1}{2a} f(\zeta)$$

$f(\zeta), -a < \zeta < a$

As $a \to 0, f(\zeta) \to f(0)$

i.e.

$$\lim_{a \to 0} \delta_a(x) \to \delta(x)$$

Note

$$\delta_\alpha(x) \equiv \sqrt{\frac{\pi}{\alpha}} e^{-\alpha x \zeta}$$

$\alpha \to 0$

$\delta_\alpha(x) \to \delta(x)$

Note

$$\tilde{\delta}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ipx} \delta(x)$$

$$= \frac{1}{\sqrt{2\pi}}$$

Note

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, e^{ipx} \tilde{\delta}(p)$$

$$\boxed{\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} dp}$$

$\delta(x-a): \int_{-\infty}^{\infty} dx\, f(x) f(x-a) = f(a)$

$\delta(x-a) \mapsto ?$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ipx}\delta(x-a) = \frac{a}{\sqrt{2\pi}}$$

$$f(x)\delta(x) = f(0)\delta(x)$$

As a distribution

Proof

$$\int_{-\infty}^{\infty} dx\,\big(f(x)\delta(x)\big)g(x) = \int_{-\infty}^{\infty} dx\, f(x)g(x)\delta(x) = f(0)g(0)$$

$$\int_{-\infty}^{\infty} dx\, f(0)\delta(x)g(x) = f(0)g(0)$$

A

Also

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

Proof

$$\int_{-\infty}^{\infty} dx f(x)\delta(ax)$$

Let $x' = \alpha x$

$$\alpha dx = dx'$$

$$dx = \frac{1}{\alpha}dx$$

IF

$$a > 0, x = \pm\infty \Rightarrow x' = \pm\infty$$

$$= \frac{1}{\alpha}\int_{-\infty}^{\infty} dx' f\left(\frac{x'}{\alpha}\right)\delta(x') = \frac{1}{\alpha}f(0)$$

Suppose f(x) is periodic $f(x+2\pi) = f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ipx}\sum c_n e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{c_n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-i(p-n)x}$$

$$= \sum_{n=-\infty}^{\infty} c_n\sqrt{2\pi}a\delta(p-n)$$

Recap

$$\int_{-\infty}^{\infty} dx\, \delta(x)f(x) = f(0)$$

$$\tilde{\delta}(o) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ipx}\,\delta(x) = \frac{1}{\sqrt{2\pi}}$$

$$\delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dp\, e^{ipx}\,\tilde{\delta}(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{ipx}$$

Example

$$Let\ f(x) = x = x * 1 \mapsto (\square)\frac{\delta}{\delta p}\delta p$$

$$\tilde{f}(p) = \frac{1}{\sqrt{\square}}$$

What does

$$\frac{d\delta(x)}{\text{dx}}$$

Mean?

Now

$$\int_{-\infty}^{\infty} dx\, f(x)\frac{d}{dx}\delta(x) = [f(x)\delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}f(x)\delta(x)dx$$

$$= f'(0)$$

Use of δ function

Result

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} \left|\tilde{f}(p)\right|^2 dp$$

RHS

$$= \int_{-\infty}^{\infty} dp \, \tilde{f}(p)\tilde{f}(p)^*$$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-ipx} f(x)$$

$$\tilde{f}^*(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' \, e^{ipx'} f^*(x')$$

Important to label dummy variables separately

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' e^{ip(x'-x)} f(x) f^*(x')$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \, f(x) f^*(x') \int_{-\infty}^{\infty} dp \, e^{ip(x'-x)}$$

$$\int_{-\infty}^{\infty} dp \, e^{ip(x'-x)} = 2\pi\delta(x'-x)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dx' f(x) f^*(x') \delta(x'-x)$$

$$= \int_{-\infty}^{\infty} dx f(x) \int_{-\infty}^{\infty} dx' f^*(x') \delta(x'-x)$$

$$\int_{-\infty}^{\infty} dx' f^*(x') \delta(x'-x) = f^*(x)$$

$$= \int_{-\infty}^{\infty} dx \, f(x) f^*(x) = LHS$$

Application: QM

13 March 2012
14:19

$$|S\rangle = \sum_{i=1}^{n} C_i \, |i\rangle$$

$$\psi_S = \sum_{i=1}^{n} c_i \psi_i$$

ψ_i =eigenstates of an operator

$$c_i = \int d^3x \, \psi_i^* \psi = \langle \psi_i \mid \psi \rangle$$

Application: Quantum mechanics

$$iC_i = (if)\langle j|i\rangle = \delta_{ij}$$

$$\int \psi_j^* \psi_i = \delta_{ij}$$

We can extend to case when we have an infinite, countable set of states

$$|s\rangle = \sum_{i=1}^{\infty} C_i \mid i\rangle$$

Suppose we move to continuous infinity

$$\mid S\rangle = \int_{-\infty}^{\infty} dx \, C_x \mid x\rangle$$

Important case: expand in terms of eigenstates of $\hat{x}$

$\mid S\rangle = \int dx \, C(x) \mid x\rangle$

$C_x \to C(x)$

$\hat{x} \mid x\rangle = x \mid x\rangle$

$\equiv \psi(x)$

We can also expand $\mid s\rangle$ in terms of eigenstates of momentum

$\mid S\rangle = \int dp \tilde{\psi}(p) \mid p\rangle$

$\hat{p} \mid p\rangle = p \mid p\rangle$

$\hat{x}, \hat{p}$ are operators

$\hat{x}\psi(x) = x\psi(x)$

$$\hat{p}\psi(x) = i\hbar \frac{\delta}{\delta x} \psi(x)$$

Momentum eigenstate is

$$\psi(x) = e^{-ip\frac{x}{\hbar}}$$

$$\hat{p}\psi(x) = i\hbar \frac{\delta}{\delta x} e^{-ip\frac{x}{\hbar}} = i\hbar - \frac{ip}{\hbar} e^{-ip\frac{x}{\hbar}}$$

$= P\psi(x)$

=Eigenstate of $\hat{p}$

Note $\psi(x)$ is NOT normalizable!

A. Put universe in a box
   - $-M \le x \le M$
   - $\psi(x)$ is normalizable
   - $\int_{-M}^{M} |\psi|^2 < \infty$
   - We can calculate "everything" at finite M, & set M to infinity at last line

B. Use wave packets as a fundamental state
   - $\psi(x)$ is a Gaussian at x_0 with width Δx
   - And we have $\psi(p)$ a Gaussian of width ΔP

C. Use δ −functions
   - Accept $\psi(x)$ being a measure rather than function
   - $\int_a^b |\psi(x)|^2$= probability of finding particle between a and b
   - $|\psi(x)|^2 \Delta x$ = probability of finding particle between x & Δx

Since $\psi(x)$ inside $\int$ perhaps a measure is OK

$$\psi_S = \sum_{i=1}^{n} c_i \psi_i$$

ψ_i =eigenstates of an operator

$$c_i = \int d^3x\, \psi_i^* \psi = \langle \psi_i \mid \psi \rangle$$

For a free particle, particle can be in state x

$$\hat{x} \mid x\rangle = x \mid x\rangle$$

$$\psi(x) \equiv c_i$$

Eigenstates of momentum

$$e^{-\frac{ipx}{\hbar}} = \psi_p(x)$$

$$\hat{p} = i\hbar \frac{\delta}{\delta x}\, \psi_p(x) = p\psi_p(x)$$

Let $\psi_{p_1}(x)$& $\psi_{p_2}(x)$ be wavefunctions which are eigenstates of momentum

Recap for

$$\psi = \sum_n c_n \psi_n$$

$$\int \psi_n^* \psi_m = \delta_{n,m}$$

$$\int dx\, \psi_{p_1}^*(x) \psi_{p_2}(x)$$

$$= \int_{-\infty}^{\infty} dx\, e^{\frac{ip_1 x}{\hbar}} e^{-\frac{ip_2 x}{\hbar}}$$

$$= \int_{-\infty}^{\infty} dx\, e^{\frac{i(p_1-p_2)x}{\hbar}}$$

$$= 2\pi \times 8\left(\frac{p_1 - p_2}{h}\right)$$

$$= 2\pi \times \hbar \times \delta(p_1 - p_2)$$

x-representation	p-representation
$\psi(x)$	$\psi'(p)$
$\hat{x}\psi = x\psi$	$\hat{p}\psi' = p\psi'$
$\hat{p}\psi = i\hbar \frac{\delta}{\delta x}\psi$	$\hat{x}\psi'(p) =?$ $\leftarrow i\hbar\, \frac{\delta}{\delta p}$
$\psi_p = e^{-\frac{ipx}{\hbar}}$ Eigenfunction of $\hbar$ Eigenvectors of x	Eigenstates of x? $\tilde{\psi}(p)$ $= \delta(p - p_0)$ $\times 2\pi\hbar$ $= \frac{\hbar}{2\pi}\delta(p_0 - p)$

$\psi(x)$&$\psi'(p)$ are related by being fourier transform of each other ($\hbar = 1$)

$$\psi'(p) = \tilde{\psi}(p)$$

Recall that

$$\psi \mapsto \tilde{\psi}$$

$$x\psi \mapsto \sim \frac{\delta}{\delta p}\tilde{\psi}$$

$$\frac{\delta \psi}{\delta x} \mapsto \sim p x \tilde{\psi}$$

For

$\psi_p = e^{\frac{ip_0 x}{\hbar}}$

What is $\tilde{\psi}(p)$

$$\psi(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{ip_0}{\hbar}x} e^{-\frac{ip}{\hbar}x}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{i(p_0-p)}{\hbar}x}$$

Similarly eigenstate of x (with value x_0) is

$$\frac{\hbar}{\sqrt{2\pi}}\delta(x-x_0) = \psi(x)$$

Example

$$f(x) = \begin{Bmatrix} \cos(k_0 x) & |x| \leq L \\ 0 & otherwise \end{Bmatrix}$$

"wave train"

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ipx} f(x)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} dx\, e^{-ipx}\cos(k_0 x)$$

Trick use

$$\cos(k_0 x) = \frac{1}{2}\left(e^{ik_0 x} + e^{-ik_0 x}\right)$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-L}^{L}\left(e^{-i(p+k_0)x} + e^{-i(p-k_0)x}\right)dx$$

$$= \frac{1}{2\sqrt{2\pi}}\left\{\left[\frac{e^{-i(p+k_0)x}}{-i(p+k_0)}\right]_{-L}^{L} + \left[\frac{e^{-i(p-k_0)x}}{-i(p-k_0)}\right]_{-L}^{L}\right\}$$

$$= \frac{1}{2\sqrt{2\pi}}\left\{\frac{e^{-i(p+k_0)L} - e^{-i(p+k_0)(-L)}}{-i(p+k_0)} + \frac{e^{-i(p-k_0)L} - e^{-i(p-k_0)(-L)}}{-i(p-k_0)}\right\}$$

Use $\frac{e^{i\alpha}-e^{-i\alpha}}{2i} = \sin\alpha$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin\left((p+k_0)L\right)}{p+k_0} + \frac{\sin\left((p-k_0)L\right)}{p-k_0}\right]$$

$$= \frac{L}{\sqrt{2\pi}} \times [\text{sinc}(p+k_0)L + \text{sinc}(p-k_0)L]$$

Function crosses axis at

$$(p-k_0)L = \pi$$

$$p - k_0 = \frac{\pi}{L}$$

$$p = k_0 + \frac{\pi}{L}$$

As L shortens pulse/fourier transform widens

Note peak at $p = -k_0 \& p = k_0$

Because

$$\cos(k_0 x) = \frac{1}{2}\left(e^{ik_0 x} = e^{-ik_0 x}\right)$$

Peaks have width $\frac{\pi}{L}$

Peaks with long width, $L \to 0$

Peaks with short widths $L \to large$

<u>Convolutions</u>

Take 2 functions $f(x), g(x)$

Convolution

$$h(x) = (f * g)x$$

$$h(x) = \int_{-\infty}^{\infty} dx' f(x') g(x - x')$$

<u>Applications</u>

$\Rightarrow^{signal} \Big|_{\text{screen f(x)}} \Big|_{g(x)} \Rightarrow$ output is some convolution of original signal

Properties

$f * g = g * x$

Proof

$$LHS = f * g(x) - \int_{-\infty}^{\infty} dx' f(x') g(x - x')$$

Let

$y = x - x'$

$dy = -dx'$

$x' = +\infty \Rightarrow y = i\infty$

$x' = x - y$

$$f * g(x) = \int_{\infty}^{-\infty} -dy\, f(x - y) g(y)$$

$$= \int_{-\infty}^{\infty} dy\, g(y) f(x - y) = g * f(x)$$

$f * \delta(x) = f$

$$(f * \delta)(x) = \int_{-\infty}^{\infty} dx' f(x') \delta(x - x') = f(x)$$

Proof

If

$h(x) = f * g$

And

$f \mapsto \tilde{f}$

$g \mapsto \tilde{g}$

$h \mapsto \tilde{h} = \sqrt{2\pi} \tilde{f}(p) \tilde{g}(p)$

Proof

$$\tilde{h}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ipx} h(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' g(x') f(x - x') e^{-ipx}$$

Reorder integrators

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' g(x') \int_{-\infty}^{\infty} dx f(x - x') e^{-ipx}$$

Let $y = x - x' \Rightarrow dy = dx, x = y + x'$

$x = \pm\infty \Rightarrow y = \pm\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' g(x') \int_{-\infty}^{\infty} dy\, f(y) e^{-ip(y + x')}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' g(x') e^{-ipx'} \int_{-\infty}^{\infty} dy\, f(x) e^{-ipy} = \sqrt{2\pi} \tilde{f}(p)$$

$$\int_{-\infty}^{\infty} dx' g(x') e^{-ipx'} = \sqrt{2\pi} \tilde{g}(p)$$

So

$\tilde{h}(p) = \sqrt{2\pi} \tilde{f}(p) \tilde{g}(p)$

Strategy to disentangle a convolution (knowing f or g)

$$f * g \to_{FT} \tilde{f}(p) = \frac{\widetilde{f * g}}{\tilde{g}(p)} \to \tilde{f}(p) \to_{FT} f(p)$$

Result

If

$f \mapsto \tilde{f}(p), g \mapsto \tilde{g}(p)$

If

$h(x) = f(x) g(x), h \mapsto \tilde{h} \Rightarrow \tilde{f} * \tilde{g}$

Suppose

$\tilde{h}(p) \equiv \tilde{f}(p) * \tilde{g}(p)$

Then

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, e^{ipx} \tilde{h}(p)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, e^{ipx} \int_{-\infty}^{\infty} dp' \tilde{f}(p') \tilde{g}(p - p')$$

Changing order of integrator, and define $q \equiv p - p'$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp' \tilde{f}(p') \int_{-\infty}^{\infty} dq\, \tilde{g}(q) e^{i(p'+q)x}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp' f(p') e^{ip'x} \int_{-\infty}^{\infty} dq\, \tilde{g}(q) e^{iqx}$$
$$\int_{-\infty}^{\infty} dp' f(p') e^{ip'x} = f(x)$$
$$\int_{-\infty}^{\infty} dq\, \tilde{g}(q) e^{iqx} = g(x)$$

So

$$h(x) = \frac{1}{\sqrt{2\pi}} f(x) g(x)$$

So

$$\tilde{h}(p) = \sqrt{2\pi} \tilde{f}(p) * \tilde{g}(p)$$

Complex analysis

20 March 2012
14:25

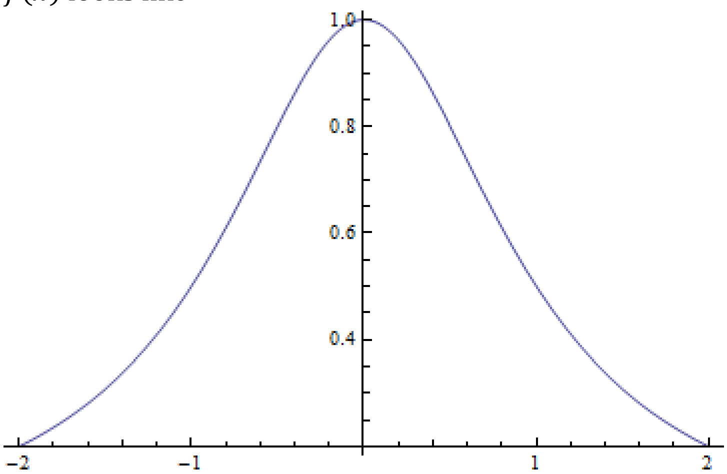
"complex valued complex functions"

Real valued function of real variable	$f\mathbb{R} \to \mathbb{R}$
Complex valued functions of real variables	$f\mathbb{R} \to \mathbb{C}$

$f(x) = f_1(x) + if_2(x)$
Useful
in waves (As a trick)
In QM $\psi(x)$

Complex valued functions of complex variables	$f\mathbb{C} \to \mathbb{C}$

If $z = x + iy$
$f(x) = f_1(x, y) + if_2(x, y)$
At present it is a useful trick/technique
Can help to understand real functions
e.g. $f(x) = \frac{1}{1+x^2}$
Consider taylor expansion
$$\left(\frac{1}{1-x} = 1 + x + x^2 + x^3\right)$$
$f(x) = 1 - x^2 + x^4 - x^6 +$
Converges $|x| < 1$
Diverges $|x| \geq 1$
$f(x)$ looks like



$f(x)$ has no bad behaviour at x=1
Consider
$f\mathbb{C} \to \mathbb{C}$
$$f(z) = \frac{1}{1+z^2}$$
$f(z)$ has a singularity at $1 + z^2 = 0$, or $z^2 = -1, z = \pm i$
If $z = iy$
$$f(x) = \frac{1}{1-y^2}$$
We can understand the behaviour of f(x) better by studying f(z)
Hot topic in particle physics
Studying f if k_1, k_2 can be complex is a remarkably useful technique

Allows us to do really really hard integrals

Complex numbers

26 March 2012
12:06

Complex number

$Z = x + iy, -\infty < x, y < \infty$

$= re^{i\theta}, 0 \le r < \infty, 0 \le \theta < 2\pi\ (-\pi < \theta \le \pi)$

$Z^* = x - iy = re^{-i\theta}$

$ZZ^* = |Z|^2 = r^2 = x^2 + y^2$

$\frac{1}{Z} = \frac{Z^*}{ZZ^*} = \frac{Z^*}{|Z|^2}, e.g. \frac{1}{i} = -i$

$Z_1Z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1) = r_1r_2e^{i(\theta_1+\theta_2)}$

$|Z_1Z_2| = |Z_1||Z_2|$

For $Z = x + iy$

$x = Real(Z) = \frac{Z + Z^*}{2}$

$y = Imagenary(x) = \frac{Z - Z^*}{2i}$

Regions on the complex plane

$\{Z: |Z - Z_0| \le a\}$

a=real number

Functions

f(z) is a complex function

$f: \mathbb{C} \to \mathbb{C}$

Limits

$\lim_{z \to z_0} f(z) = \omega_0$

Means as z gets closer to x_0 from any direction, $f(z)$ gets close to ω_0

$\forall \epsilon, \exists a$ st $|f(z) - \omega_0| < \epsilon$ for $z \in \{z: |z - z_0| < a\}$

Differentiation

Consider

$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

If this exists then f(z) is differentiable/Holomorphic/analytic

At z_0 we denote the limit by $f'(z_0)$

e.g. any polynomial is holomorphic $f(z) = a_0 + a_1z - \cdots a_nz^n$

Take something that is not holomorphic

e.g. f(z)=Re(z)

$\Rightarrow Consider$

$\frac{f(z) - f(0)}{z - 0}$

Consider the limit along real axis, z=x

$\frac{f(z) - f(0)}{z - 0} = \frac{x - 0}{x - 0} = 1$

Consider the limit along imaginary axis z=iy

$\frac{f(z) - f(0)}{z - 0} = \frac{0}{iy} = 0$

$\Rightarrow$ two different limits imply function is not holomorphic

Also,

$Im(z) = \frac{z - Re(z)}{i} = -iz + iRe(z)$

Also, Z^* is not holomorphic

$\left(Re(z) = \frac{1}{2}(z + z^*)\right)$

$|z|^2 = zz^*$ is not holomorphic

E.G. $Z^n = f(z)$

$\frac{f(z) - f(z_0)}{z - z_0} = \frac{z^n - z_0^n}{z - z_0} = \frac{(z - z_0)}{z - z_0}(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1})$

$= z^{n-1} + z^{n-2}z_0 + \cdots + z^{n-1}z_0$

So limit

$= nz_0^{n-1}$

e.g. complex function

$e^z = e^xe^{iy}$

$e^{z'} = e^z\ \left(\frac{df}{dz} = f\right)$

e.g.

$f(z) = \frac{1}{z}$

=holomorphic except at z=0

$f' = -\frac{1}{z^2}$

= singular at z=0

="pole" at z=0

Rational function

$f(z) = \frac{\sum_{n=0}^{r_2} a_nz^n}{\sum_{n=1}^{r_1} b_nz^n} = \frac{(a_0 + a_1z + \cdots + a_nz^n)}{(b_0 + b_1z + \cdots + b_nz^n)}$

Singularities occur when

$\sum b_nz^n = 0$

Or

$b_{r_1}(z - z_1)(z - z_2)(z - z_3) \ldots (z - z_r)$

e.g. $z^2 + 1 = (z + i)(z - i)$

NB. Real polynomial

$$P(x) = \prod_{i=1}^{m} (x - x_i)x_0 \times \prod_{j=1}^{n} (x^2 + bx + c)$$

IF z_i appears once, it is a "simple pole"

e.g.

$$f(z) = \frac{1}{(z-1)(z-2)(z-3)}$$

Simple poles at $z = 1,2,3$

$$\frac{1}{(z+1)^2(z-1)}$$

Simple pole at z=1

But pole of degree 2 at z=-1

e.g. Log(z)

Define this by

$e^{\log z} = z$

Let $z = re^{i\theta} = x + iy$

$\log(z) = f + ig$

$e^{f+ig} = re^{i\theta}$

Pr

$e^f = r$

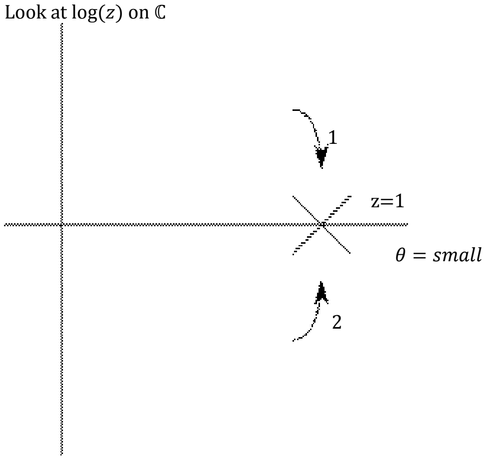
$e^{ig} = e^{i\theta}$

$f = \log|z|$

$g = \arg(z)$

$Log\,(z) = \log|z| + i\arg(z)$

Look at $\log(z)$ on $\mathbb{C}$



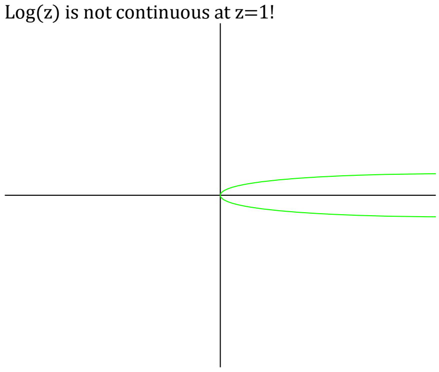
1. Approach from above

   $\log z = i\theta \to 0$

2. Approach from below

   $\log z = i\theta \to 2\pi i$

Log(z) is not continuous at z=1!



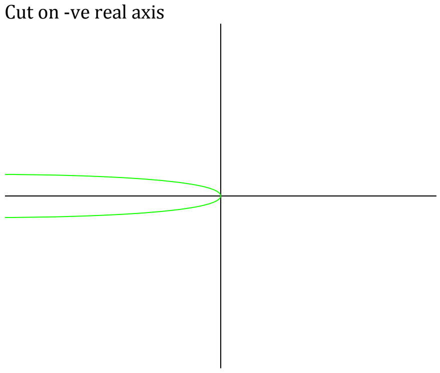
Not continuous at any point on real +ve axis

Log(z) is not good on real axis

We could choose $-\pi < \theta' < \pi$

$\log' z = \log(|z|) + i\theta'$

Cut on -ve real axis



Functions with cuts are tricky

e.g. $f(z) = \sqrt{z}$

$z = re^{i\theta}, 0 \le \theta \le 2\pi$

$\Rightarrow f(z) = \sqrt{r}e^{i\frac{\theta}{2}}$

Has a cut on real axis

We can change position of cut, BUT not its existence

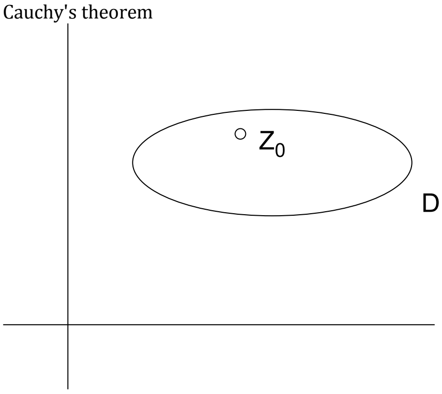
e.g.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Give holomorphic functions everywhere

Cauchy's theorem

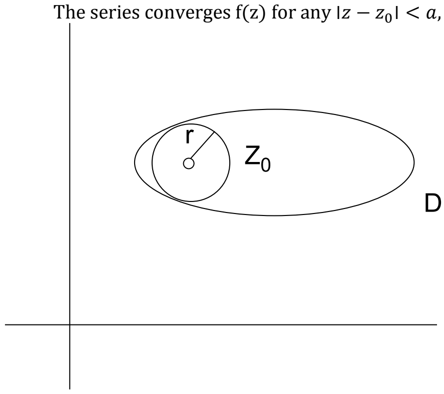


f(z) is holomorphic on D & $z_0 \subset D$

Then f(z) can be expanded

$$f(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n$$

The series converges f(z) for any $|z - z_0| < a$, where a is largest disc fitting into D



=Taylor series converges for z in disc

e.g

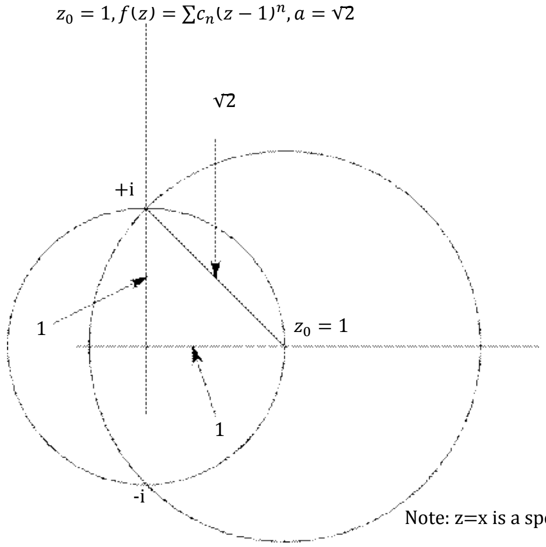
$$f(z) = \frac{1}{1 + z^2}$$

Holomorphic except at $z = \pm i$

For $z_0 = 0, a = 1$ & $radius\ of\ convergence = 1$

For

$z_0 = 1, f(z) = \sum c_n (z-1)^n, a = \sqrt{2}$



Note: z=x is a special case

A curve, γ, is a continuous mapping from [a,b] to $\mathbb{C}$, $z = \gamma(t), t \in [a, b]$

$\gamma(t)$ is the parameterisation, $\gamma \& \gamma'$ are same if set of points $\{\gamma(t)\} = \{\gamma'(t)\}$

e.g.

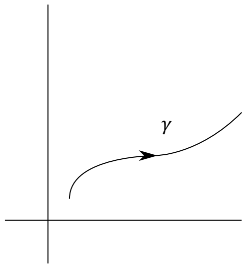
$\gamma_1(t) = 1 + it, t \in [0,1]$

$\gamma_2(t) = 1 + it^2, t \in [0,1]$

= these are two parameterisations of the same curve

e.g. $\gamma(t) = 2e^{it}, t \in [0,2\pi]$

Contour integrals



$\int_\gamma f(z)dz \equiv depends\ upon\ z\ \&\ \gamma$

$\int_\gamma f(z)dz = \int_a^b dt\, f\left(z(\gamma(t))\right)\gamma'(t)$

e.g.

$f(z) = (1+z^2), \{\gamma(t) = 1+it, t \in [0,1]\}$

$\int_\gamma f(z)dz = \int_0^1 (1+(1+it)^2).idt$

$= i\int_0^1 dt\,[1+1+2it-t^2]$

$= i\left|2t = \frac{2it^2}{2} - \frac{t^3}{3}\right|_0^1 = i\left|2+i-\frac{1}{3}\right| = -1+\frac{5}{3}i$

$= \int_\gamma f(z)dz$

Is parameterization invariant

e.g. $f(z) = 1+z^2\{\gamma(t) = 1+it^2, t \in [0,1]\}$

$\int_0^1 (1+(1+it^2)^2.2it\,dt$

$2i\int_0^1 t(1+1+2it^2-t^4)$

$= 2i\left|\frac{2t^2}{2} + \frac{2it^4}{4} - \frac{t^5}{5}\right|_0^1$

$= 2i - 1 - \frac{1}{3}i = -1+\frac{5}{3}i$

e.g.

$\gamma(t) = Re^{it}, 0 \le t \le 2\pi$

=circle of radius R

="anticlockwise" means +ve direction in θ

$f(z) = z^n, n \ge 0\ integer$

$\int_\gamma f(z)dz = \int_0^{2\pi} dt(Re^{it})iRe^{it} = iR^{n+1}\int_0^{2\pi} e^{i(n+1)t}dt = iR^{n+1}\left|\frac{e^{i(n+1)t}}{1(n+1)}\right|_0^{2\pi}$

$= \frac{iR^{n+1}}{i(n+1)}[1-1] = 0$

$f(z) = \frac{1}{z^n}, n \ne 1$

$\int_\gamma f(z)dz = \int_0^{2\backslash pi} dt\,\left(\frac{e^{-int}}{R}\right)iRe^{it} = 0$

$f(z) = \frac{1}{z}$

$\int_\gamma f(z)dz = \int_0^1 dt\,\frac{1}{Re^{it}}(iRe^{it}) = i\int_0^{2\pi} dt\,2\pi i$

$\int_\gamma f(z)dz = 2\pi i\,\delta_{n,-1}$

Function gives nothing unless n=-1

Cauchy's integral theorem

$f(z)$ holomorphic on D

γ is in D closed simple contour

$\int_\gamma f(z)dz = 0$

- Proof (Not a proof, but reason)

  A small circular region

  $\int_\gamma f(z)dz = 0$

  $f(z) = z^n, n = any + ve\ integer$

  Trivial extension to a polynomial

  For a general holomorphic function

  $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$

  In a small region

  $f_N(z) = \sum_{n=0}^{N} c_n(z-z_0)^n$

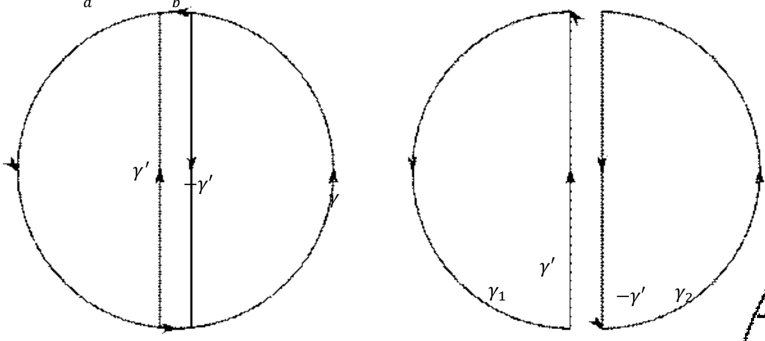
  satisfies

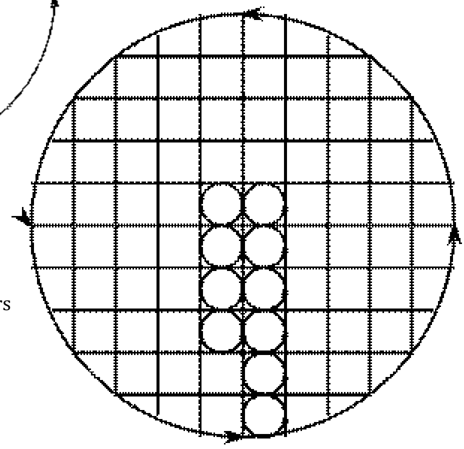
  $\int_\gamma f_N(z)dz = 0$

- Note: γ has a direction. Contour reversed is $-\gamma$

  $\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz$

$\Rightarrow \int_a^b dt = -\int_b^a dt$





$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_1} f(z)dz$

$\int_\gamma f(z)dz = \sum_{CEUS} \int_{\gamma_1} f(z)dz$

Since cauchy's works for γ_i, it works for γ- contour can be split into several small contours

<u>Cauchy's residue theorem</u>

γ closed, simple, anticlockwise [+ve direction]

$\int_\gamma f(z)dz = 2\pi i \times \sum_{z_i \, inside \, \gamma} Res(f, z_i)$

$Res(f, z_i) \equiv residue \; of \; f \; at \; z_i$

$Res(f, z_i) = \lim_{z \to z_i} (z - z_i) f(z)$

$Res(f, z_i) \equiv$"size" of singularity

Example

$f(z) = \frac{1}{a^2 + z^2}$

A +ve real number

f(z) holomorphic except $a^2 + z^2 = 0$

$z^2 = -a^2$

$z = \pm i a^2$

$Res(f, ia) = \lim_{z \to ia} (z - ia) \frac{1}{z^2 + a^2}$

$\lim_{z \to ia} (z - ia) \frac{1}{(z - ia)(z + ia)} = \lim_{z \to ia} \frac{1}{(z + ia)} = \frac{1}{2ia} = \frac{-i}{2a}$

$Res(f, -ia) = \lim_{z \to ia} (z + ia) \frac{1}{z^2 + a^2}$

$\lim_{z \to ia} (z + ia) \frac{1}{(z - ia)(z + ia)} = \lim_{z \to ia} \frac{1}{(z - ia)} = -\frac{1}{2ia} = \frac{i}{2a}$

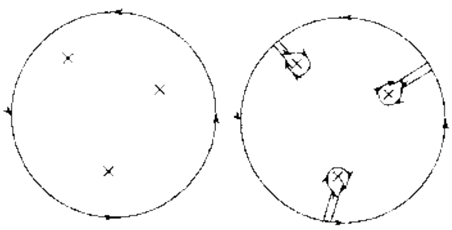
$\int_{\gamma_1} f(z)dz = 2\pi i \, Res(f, ia)$

$2\pi i * -\frac{i}{2a} = \frac{\pi}{a}$

$\int_{\gamma_2} f(z)dz = 2\pi i [Res(f, ia) + Res(f, -ia)] = 0$

$\int_{\gamma_3} f(z)dz = 0$

Proof

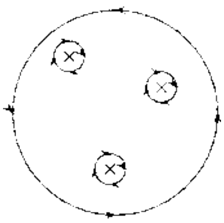


$\times \; = z_i = z_{1,2,3}$

Consider $\gamma' \equiv$ new contour

$\int_{\gamma'} f(z)dz = 0$

$\gamma' \equiv$



Near z_i

$f(z) \approx \frac{Res(f, z_i)}{z - z_i}_{small} \; as \; z \to z_i$

& we know that

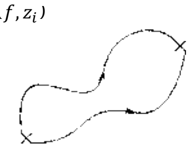
$\int_\gamma \frac{1}{dz} dz * 2\pi i$

for γ a positive circle of radius R

$$0 = \int_{\gamma} f(z)dz = \int_{\gamma} f(z)dz - 2\pi i \sum Res(f, z_i)$$

$$\int_{\gamma} f(z)dz = 2\pi i \sum_{i} Res(f, z_i)$$

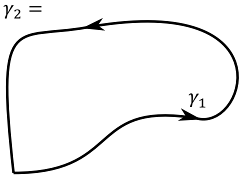
Note: we can distort contours



$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

If f(z) is holomorphic on shaded region

$\gamma_1 \to \gamma_2 =$



And $\int_{\gamma_1 - \gamma_2} f(z)dz = 0$

Example:

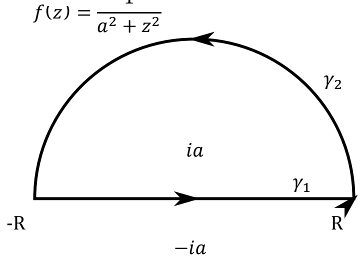
Consider

$$I = \int_{-\infty}^{\infty} dx \frac{1}{a^2 + x^2}$$

We must choose f(z) & γ

Choose

$$f(z) = \frac{1}{a^2 + z^2}$$



$$\gamma = \gamma_1 + \gamma_2$$

Note

$$\int_{\gamma_1} f(z)dz$$

$$\gamma_1 = \begin{cases} z = x, & -R \le x \le R \\ \gamma' = 1 \end{cases}$$

$$= \int_{-R}^{R} \frac{dx}{a^2 + x^2} = I_R$$

-we want the limit as $R \to \infty$

$$\int_{\gamma} f(z)dz = 2\pi i \sum_{i} Res(f, z_1) = 2\pi i \, Res(f, z = i, a), R > a$$

$$= 2\pi i * - \frac{i}{2a} = \frac{\pi}{a}$$

So

$$I_R + \int_{\gamma_2} f(z)dz = \frac{\pi}{a}$$

Note: IF

$$\int_{\gamma_2} f(z)dz \to 0 \; as \; R \to \infty$$

Then

$$I_R \to \frac{\pi}{a} = \int_{-\infty}^{\infty} dz \frac{1}{x^2 + a^2}$$

Theorem

If

$|f(z)| \le M \; on \; \gamma$

$$\left| \int_{\gamma} f(z)dz \right| \le M \times L$$

$L \equiv length \; \gamma$

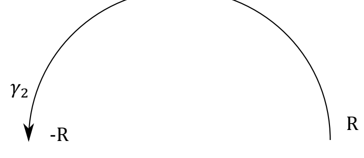
$L \equiv \int_{a}^{b} dt |\gamma'(t)| \equiv$ normal definition of length

This is just

$$|a_1 + a_2 + a_3| \le |a_1| + |a_2| + |a_3|$$

Theorem

$\gamma_2 = z = Re^{i\theta}, 0 \le \theta \le \pi$



On $\gamma_2 |z| = R$

On γ_2

$$f(z) = \frac{1}{a^2 + z^2}$$

$$|f(z)| = \frac{1}{|a^2 + z^2|} \leq \frac{1}{|z|^2 - |a|^2} = \frac{1}{R^2 - a^2}$$

$$\left| \int_{\gamma_2} f(z)dz \right| \leq \frac{\pi R}{R^2 - a^2}$$

$R \to \infty$

$this \to 0$

So

$$\lim_{R\to\infty} I_R = \frac{\pi}{a}$$

29 March 2012
12:11

This technique works

$$\int_{-\infty}^{\infty} \frac{p^m(x)}{p^n(x)} dx$$

Provided $n-, \geq 2$

$$\left[\text{Note } \int^{\infty} \frac{p^m(x)}{p^n(x)} \sim \int^{\infty} \frac{1}{x^{n-m}} \text{ converges only if } n-m > 1\right]$$

$$[\text{Note } \int_1^A \frac{dx}{x} = [\ln x]_1^A = \ln A \to \infty \; as \; A \to \infty$$

2) $p^n(z) = 0$ only for z not purely real

⇒in this case

$$\int_{-\infty}^{\infty} \frac{1}{x} dx$$

Is also not well defined

Example

What is fourier transform of

$$\frac{1}{x^2 + a^2}?$$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{e^{-ipx}}{x^2 + a^2}$$

Choose

$$g(z) = \frac{e^{-ipz}}{z^2 + a^2}$$

$$\int_{\gamma} g(z) dz = 2\pi i \sum_{\mathrm{z_i \; in} \; \gamma} Res(f, z_i)$$

g(z) has poles at $z = \pm ia$

$$Res(f, ia) = \lim_{z \to ia} z_i$$

$$= \lim_{z \to ia} (z - ia) \frac{e^{-ipz}}{(z - ia)(z + ia)}$$

$$= \frac{e^{-ip.ia}}{2ia} = -\frac{i}{2a} e^{pa}$$

$$Res(f, z = -ia) = \lim_{z \to ia} \frac{(z + ia) e^{-ipz}}{(z - ia)(z + ia)} = \frac{e^{-ip.-ia}}{-2ia} = \frac{i}{2a} e^{-pa}$$

Look at

$$\int_{\gamma_2} f(z) dz \,, z = Re^{i\theta} = R\cos\theta + iR\sin\theta$$

$$f(z) = \frac{e^{ipz}}{z^2 + a^2} = \frac{e^{ipR\cos\theta + Rp\sin\theta}}{z^2 + a^2}$$

$$|f(z)| = \frac{e^{Rp\sin\theta}}{|z^2 + a^2|} \leq \frac{e^{Rp\sin\theta}}{R^2 - a^2} \leq \frac{1}{R^2 - a^2}, IF \; p < 0$$

Choose

$$IF \; p < 0, \left| \int_{\gamma_2} f(z) dz \right| \leq \frac{\pi L}{R^2 - a^2} \to 0 \; as \; R \to \infty$$

So

$$\int_{\gamma} f(z) dz = 2\pi i \times -\frac{i}{2a} e^{Pa} = \frac{\pi}{a} e^{Pa}$$

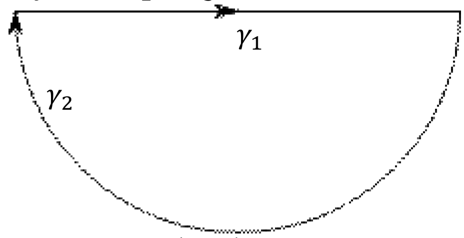
So

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \times \frac{\pi}{a} e^{Pa}$$

If P<0

$P > 0$

Try $\gamma' = \gamma_1 + \gamma_3$



$z = Re^{i(2\pi - t)}, t \in (0, \pi)$

$$\int_{\gamma} f(z)dz = -2\pi i \, Res(f, z = -ia)$$

$-$ sign since γ' is clockwise (negative)

$$= -2\pi i . \frac{1}{2a} e^{-pa} = \frac{\pi}{a} e^{-pa}$$

On $\gamma_3, z = R\cos(2\pi - t) + iR\sin(2\pi - t)$

$\left|e^{-ipz}\right| = e^{pR\sin(2\pi - t)} \leq 1$

So

$$\left|\frac{e^{-ipz}}{z^2 + a^2}\right| \leq \frac{1}{R^2 - a^2}$$

So

$$\int_{\gamma_3 \to 0} f(z)dz$$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \frac{\pi}{a} e^{-pa}, p > 0$$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \frac{\pi}{a} e^{pa}, p < 0$$

So

$$\boxed{\tilde{f}(p) = \sqrt{\frac{\pi}{2}} \times \frac{e^{-|p|a}}{a}}$$

e.g.6

$$f(x) = \frac{\cos(x)}{x^2 + a^2}$$

$$f(z) = \frac{e^{iz} + e^{-iz}}{z^2 + a^2} = f_1(z) + f_2(z)$$

$$f_1(z) = \frac{e^{iz}}{z^2 + a^2}$$

$$f_2(z) = \frac{e^{-iz}}{z^2 + a^2}$$

Example

$$\int_0^{2\pi} d\theta F(\cos\theta, \sin\theta)$$

e.g.

$$\int_0^{2\pi} \frac{dx}{2 + \cos A}$$

We want

$$\int_{\gamma} f(z)dz = I$$

Choose γ

Circular contour of r=1

$z = e^{i\theta}, 0 \leq \theta \leq 2\pi$

ON contour

$$\cos A = \frac{\left(e^{i\theta} + e^{-i\theta}\right)}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right), \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Note

$$\gamma'(\theta)d\theta = ie^{i\theta}d\theta$$

So

$$\int \frac{dz}{z} \times (\square) \Rightarrow \int \frac{d\theta ie^{i\theta}}{e^{i\theta}} \times (\square) = i\int d\theta(\square)$$

So

$$\int_\gamma \frac{dz}{iz} F\begin{pmatrix} \cos A \Rightarrow \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin A \Rightarrow \frac{1}{2}\left(z - \frac{1}{z}\right) \end{pmatrix}$$

$$\Rightarrow \int_0^{2\pi} d\theta F(\cos\theta, \sin\theta)$$

Eg

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$$

γ circle radius 1

$$f(z) = \frac{1}{iz} \frac{1}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} = -i \frac{1}{2z + \frac{1}{2}z^2 + \frac{1}{2}}$$

$$= -\frac{2i}{4z + z^2 + 1}$$

Ex1

$$\int_\gamma f(z) = 2\pi i \sum Res(f, z_i)$$

f(z) has poles

$$z^2 + 4z + 1 = 0$$
$$(z+2)^2 - 4 + 1 = 0$$
$$(z+2)^2 = 3$$
$$z = -2 \pm \sqrt{3}$$

$$\int_\gamma f(z)dz = 2\pi i Res\left(f, z = -2 + \sqrt{3}\right)$$

Res

$$= \lim_{z \to -2+\sqrt{3}} \left(z - \left(-2 + \sqrt{3}\right)\right) \frac{-2i}{\left(z - \left(-2 + \sqrt{3}\right)\right)\left(z - \left(-2 - \sqrt{3}\right)\right)}$$

$$= -\frac{2i}{-2 + \sqrt{3} - \left(-2 - \sqrt{3}\right)} = -\frac{2i}{2\sqrt{3}} = -\frac{i}{\sqrt{3}}$$

$$\int f(z)dz = 2\pi i \times -\frac{i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

Complex analysis

23 April 2012
12:01

1. Expnd theorem to case where f(x) has poles beyond simple,

   Recall

   $$\int_\gamma dz\, f(z) = 2\pi i \sum_{Z_i inside\, \gamma} Res(f, Z_i)$$

   $$Res(f, Z_i) \equiv \lim_{z\to z_i} (z - z_i) f(z)$$

   But

   $$f(z) = \frac{1}{(z-1)^2(z^2+1)}$$

   $\Rightarrow$ simple poles $z = \pm i$

   $\Rightarrow$ but non simple pole at $z = 1$

   $f(z)$ is said to have a pole "of order p" at z_0 if

   $$(z - z_0)^p f(x)$$

   Is holomorphic at $z = z_0 \& p$ is lowest such value

   Simple pole is a pole of order 1

   &

   $$f(z) = \frac{1}{(z-1)^2(z^2+1)}$$

   Has a pole of order 2 at z=1

   If f(z) has a pole of order P at $z = z_0$ then we can perform a "lavrant expansion"

   $$f(z) = \frac{a_{-p}}{(z-z_0)^p} + \frac{a_{-p+1}}{(z-z_0)^{p-1}} + \cdots + \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

   Proof

   Let

   $$g(z) = (z-z_0)^p f(z) \Rightarrow g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n = c_0 + c_1(z-z_0) + \cdots$$

   $$\Rightarrow f(z) = \frac{g(z)}{(z-z_0)^p} = \frac{c_0}{(z-z_0)^p} + \cdots$$

   If f(z) has a pole of order p at $z_c = z_0$ we still use Cauchy's residue theorem but $Res(f, z_1) \Rightarrow a_{-1}$

   Proof

   Recall

   $$\int_\gamma dz\, z^n = \begin{matrix} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{matrix}$$

   $\gamma \Rightarrow$ circular contour of radius R

   $\Rightarrow Follows$

   E6

   $$f(z) = \frac{1}{(z-1)^2(z^2+1)} = \frac{1}{(z-1)^2(z-i)(z+i)}$$

   Let

   $$g(z) = (z-1)^2 f(z) = \frac{1}{(z-i)(z+i)}$$

   Let

   $$z = 1 + \Delta \; (\Delta \equiv z - 1)$$

   $$g(z) = \frac{1}{(1-i+\Delta)(1+i+\Delta)}$$

   $$\left[\frac{1}{1+\alpha} = 1 - \alpha + \alpha^2 + \cdots\right]$$

   $$= \frac{1}{(1-i)\left(1+\frac{\Delta}{1-i}\right)} * \frac{1}{(1+i)\left(1+\frac{\Delta}{1+i}\right)}$$

$$= \frac{1}{(1+i)(1-i)}\left[-\frac{\Delta}{1-i}\right]\left[1-\frac{\Delta}{1+i}\right]$$
$$= \frac{1}{2}\left[1-\Delta\left[\frac{1}{1-i}+\frac{1}{1+i}\right]+\cdots\right]$$
$$= \frac{1}{2}\left[1+\Delta\left[\frac{(1+i)+(1-i)}{(1-i)(1+i)}\right]+O(\Delta^2)\right]$$
$$= \frac{1}{2}[1+\Delta\frac{2}{2}+\cdots = \frac{1}{2}[1+\Delta+O(\Delta^2)]$$
$$f(z) = \frac{g(x)}{(z-1)^2} = \frac{1}{2(z-1)^2}+\frac{1}{2(z-1)}+\cdots \Rightarrow a_{-1} = \frac{1}{2}$$

2 suppose

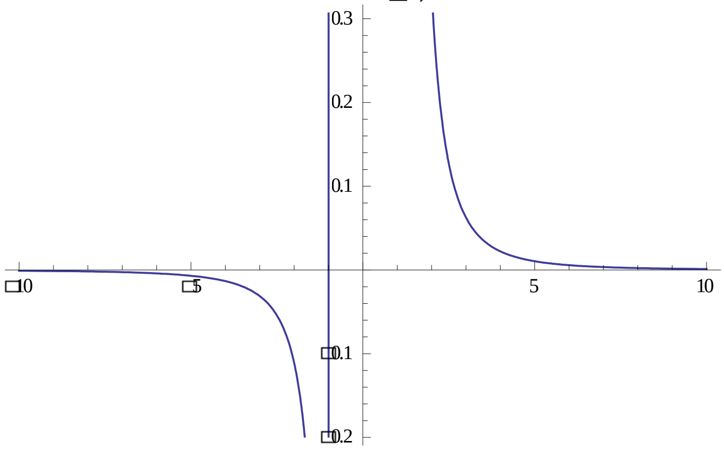
$$\int_{-\infty}^{\infty} f(x)dx$$

Suppose f(z) has a pole on real axis

e.g.

$$f(x) = \frac{1}{(x-1)(x^2+1)}$$
$$\Rightarrow f(z) = \frac{1}{(z-1)(z^2+1)}$$

⇒Poles of order 1 at $z = \pm i, z = 1$



Does it make sense to integrate it?

$$\tilde{I} = \lim_{\epsilon\to 0}\int_{-\infty} ⬚$$

****

Consider

$$I_{R,\epsilon} = \int_{-R}^{x-\epsilon} ⬚ + \int_{x+\epsilon}^{R} f(x)dx$$

& consider $\gamma+\gamma_1+\gamma_2+\gamma_3$

*****

$$\int_{\gamma_1+\gamma_2+\gamma_3} f(z)dz$$

Receives no contribution from x_0

Assume

$$\int_{\gamma_2} f(z) \to 0 \; as \; R \to \infty$$

*****

$$\int_{\gamma}\frac{dz}{z} = \int_{\theta_1}^{\theta_2} i\frac{Re^{i\theta k}d\theta}{Re^{i\theta}} = i(\theta_2-\theta_1)$$

As

$\epsilon \to 0$

$$\int_{\gamma_3} dzf(z) \to -\pi i Res(f, z_0)$$

$$\Rightarrow \int_{\gamma_1+\gamma_2+\gamma_3} \square \text{ has no contruibution}$$

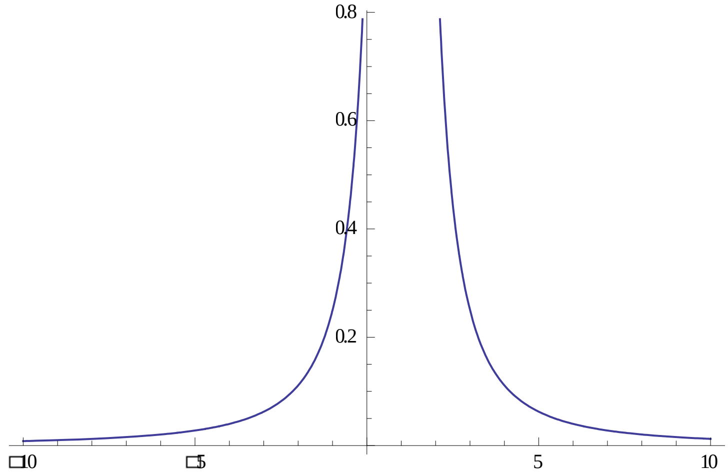
$$\Rightarrow \int_{\gamma_1} \square \text{ has a contrubution of } \pi_i Res(f, z_0)$$

This requires simple poles

*****

-Reason

$$\int dx \frac{1}{(z-1)^2}$$



No sensible definition

3

$$\int_0^\infty dx\, f(x)$$

These are much tougher

*****

- $If\ f(x) = f(-x) \Rightarrow \int_0^\infty dx\, f(x) = \frac{1}{2}\int_{-\infty}^\infty dx\, f(x)$

e.g.

$$I = \int_0^\infty \left(\frac{dx}{x^3 + a^3}\right)$$

Not even

Use

$\gamma(\theta)$

***

$On\ \gamma_3$

$$z = e^{i\theta_0}x, +R \le x \le 0$$

$$dz = e^{i\theta} dx$$

$$\int_{\gamma_3} dz f(z) = \int_{+R}^0 dx \frac{e^{i\theta_0}}{(e^{i\theta})^3 x^3 + a}$$

Choose theta such that

$$\left(e^{i\theta_0}\right)^3 = 1$$

$$\int_{+R} f(z)dz = -\int_0^R dx \frac{e^{i\theta_0}}{x^3 + a^3} = e^{i\theta_0}\int_{\gamma_1} f(z)dz$$

Choose

$$\theta_0 = \frac{2\pi}{3}$$

$\Rightarrow$

****

Poles when $z^3 + a^3 = 0$

$(z-a)(z^2 + za + a^2) = 0 \Rightarrow z = a$

Or

$$z = \frac{-a \pm \sqrt{a^2 - 4a^2}}{2} = \frac{\left(-a \pm \sqrt{3}ia\right)}{2}$$

Example of "cunning contour integration

2011-May/June exam

30 April 2012
12:04

<u>A1:</u> f(t) for $t \equiv t + 2\pi T$

$$x_{maths} = \frac{t}{T}, x_{maths} \equiv x_{maths} + 2\pi$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{t}{T}\right) + b_n \sin\left(n\frac{t}{T}\right)$$

$$dx_{maths} = \frac{dt}{T}, x_{maths} = \pi, t = \pi T$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi T}^{\pi T} \frac{dt}{T} f(t)$$

$$a_n = \frac{1}{\pi T}\int_{-\pi T}^{\pi T} dt\, f(t) \cos\left(n\frac{t}{T}\right)$$

$$b_n = \frac{1}{\pi T}\int_{-\pi T}^{\pi T} dt\, f(t) \sin\left(n\frac{t}{T}\right)$$

<u>A2:</u> let $g(x) = f(x-a)$

Then

$$\tilde{g}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ipx} f(x-a)$$

Let $y = x - a$

$$dy = dx$$

$$x = \pm\infty \Rightarrow y = \pm\infty$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, e^{-ip(y+a)} f(y)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, e^{-ipy} e^{-ipa} f(y)$$

$$= e^{-ipa} \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, e^{-ipy} f(y)$$

$$= e^{-ipa} \tilde{f}(p)$$

<u>A3:</u>

$$\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0)$$

Consider

$$\int_{-\infty}^{\infty} dx\, \delta(\alpha x) f(x)$$

Let

$$y = \alpha x, dy = \alpha dx$$

$$x = \pm\infty \Rightarrow y = \pm\infty, \alpha a > 0$$

$$\Rightarrow y = \mp\infty \quad \alpha < 0$$

$$\frac{1}{\alpha}\int_{-\infty}^{\infty} dy\, \delta(y)$$

''''''''''''''''''''''''''''

<u>A4:</u> if f(x) is holomorphic/analytic/or (in this context) differentiateable on a region, except at points Zi

Then

$$\int_{\gamma} f(z)dz = 2\pi i \sum_{z_i \text{ inside } \gamma} Res(f, z_i)$$

γ is a (simple) anticlockwise closed contour in the region

f(z) holomorphic except when

$$z^2 + 16 = 0$$

$$z^2 = -16$$
$$z = \pm 4i$$
$$Res(f, 4i) = \lim_{z\to 4i}(z-4i)\frac{z}{(z-4i)(z+4i)} = \frac{4i}{8i} = \frac{1}{2}$$
$$Res(f, -4i) = (z+4i)\frac{z}{(z-4i)(z+4i)} = \frac{1}{2}$$
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B1:

$$f(x) = \begin{cases} A & -\frac{1}{A} < x < \frac{1}{A} \\ 0 & otherwise \end{cases}$$

f(x) is an even function

$$\Rightarrow b_n = 0$$
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx f(x)$$
$$= \frac{1}{2\pi}\int_{-\frac{1}{A}}^{\frac{1}{A}} dx\, A$$
$$= \frac{1}{2\pi}[Ax]_{-\frac{1}{A}}^{\frac{1}{A}} = \frac{1}{2\pi}\left[A\frac{1}{A} - A\ \frac{-1}{A}\right] = \pi$$
$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$
$$= \frac{1}{2\pi}\int_{-\frac{1}{A}}^{\frac{1}{A}} A\cos nx\, dx$$
$$= \frac{1}{2\pi}\left[\frac{A\sin nx}{n}\right]_{-\frac{1}{A}}^{\frac{1}{A}}$$
$$= \frac{1}{2\pi}\left[\frac{A}{n}\sin\frac{n}{A} - \frac{A}{n}\sin\frac{-n}{A}\right] = \frac{1}{\pi}\frac{A}{n}\sin\left(\frac{n}{A}\right)$$

f(x)=x periodic in -$\pi < x < \pi$

Integrate by parts

$$\int_a^b fg' = [fg]_a^b - \int_a^b f'g$$
$$f = x \qquad g' = \sin nx$$
$$f' = 1 \quad g = -\frac{\cos nx}{n}$$
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$$= -\frac{2}{n}\cos n\pi = b_n$$

What does the fourier series sum to at the point x=/A?

Series sum to

$$\frac{1}{2}(f(x^+) + f(x^-))$$
$$= \frac{1}{2}\left(\frac{1}{A} + 0\right) = \frac{1}{2A}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, e^{-inx} f(x)$$
$$C_n^* = C_{-n}$$

B2:

Consider the function

$$f(x) = \begin{cases} a^{-1} & -a < x < a \\ 0 & |x| > a \end{cases}$$

Evaluate the Fourier transform $\tilde{f}(p)$ of the function. Plot a graph of $\tilde{f}(p)$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ipx} f(x) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} dx\, e^{-ipa}\frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{a}\int_{-a}^{a} dx\, e^{-ipx} = \frac{1}{\sqrt{2\pi}}\frac{1}{a}\int_{-a}^{a} dx\, e^{-ipx} = \frac{1}{\sqrt{2\pi}}\frac{1}{a}\left[\frac{e^{-ipx}}{-ip}\right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}}\frac{1}{a}\left[\frac{e^{-ipa} - e^{ipa}}{-ip}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{pa}\left[\frac{e^{ipa} - e^{-ipa}}{i}\right] = \frac{1}{\sqrt{2\pi}}\frac{2}{pa}\sin(pa)$$

Consider the function

$$g(x) = \begin{cases} a^{-} & -a < (x-d) < a \\ a^{-1} & -a < (x+d) < a \\ 0 & otherwise \end{cases}$$

$$g(x) = f(x-d) + f(x+d)$$

Look at A2- v similar problem

$$\tilde{g}(p) = \tilde{f}(p)e^{ipd} + \tilde{f}(p)e^{-ipd} = \tilde{f}(p)\left[e^{ipd} + e^{-ipd}\right]$$

$$= 2\cos(pd)\,\tilde{f}(p) = \frac{4}{\sqrt{2\pi}}\cos(pd)\frac{\sin(pa)}{pa}$$

The convolution of two functions f(x) and g(x) is defined as

$$f * g(x) = \int_{-\infty}^{\infty} dx' f(x')g(x-x')$$

Show that the fourier transform of $f * g(x)$ is given by

$$\widetilde{f * g}(p) = \sqrt{2\pi}\tilde{f}(p)\tilde{g}(p)$$

Let $h = f * g$

$$\tilde{h}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f * g(x)e^{-ipx}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx e^{-ipx}\int_{-\infty}^{\infty} dx' f(x')g(x-x')$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \int dx' e^{-ipx} f(x')g(x-x')$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx' f(x')\int_{-\infty}^{\infty} dx e^{-ipx} g(x-x')$$

Let y=x-x', dx=dy, $x = \pm\infty \Rightarrow\ y = \pm\infty$

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B3

If γ(t) is a contour

$$\int dt\, \gamma(t) f(\gamma(t)) \equiv \int_{\gamma} f(z)dz$$

$$\mathbb{C}: z = Re^{i\theta}, 0 \le \theta \le 2\pi$$

$$\frac{dx}{d\theta} = \gamma'(\theta = iRe^{i\theta}$$

$$\int dx\, f(z) = \int_{0}^{2\pi} iRe^{i\theta}d\theta\frac{1}{(Re^{i\theta})^n}$$

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For b)

$$\int_c f(z)dz = \sum_{n=-N}^{\infty} a_n\int_c dzz^n$$

Vanishes unless n=-1, $n = -1 \to 2\pi i = 2\pi i a_{-1}$

C)

$$I = \int_{-\infty}^{\infty} dx \ \frac{1}{1+x^2}$$

Use cauchy's theorem on γ

$\gamma 1, z = x, -R \leq x \leq R$

$\gamma_2, z = Re^{i\theta}, 0 \leq \theta \leq \pi$

Let

$$f(z) = \frac{1}{1+x^2}$$

Then

$$\int_{\gamma_1+\gamma_2} f(z)dz = 2\pi i \{Res(f, z_i)\}_{z_i inside\ \gamma}$$

$$f(z) = \frac{1}{1+z^2}$$

Poles when $1 + z^2 = 0$

$z^2 = -1$

$z = \pm i$

$$Res(f, z = i) = \lim_{z\to i}(z-1)\frac{1}{(z+1)(z-1)} = \frac{1}{2i} = -\frac{i}{2}$$

$$\int_{\gamma_1+\gamma_2} f(z) = 2\pi i \times -\frac{i}{2} = \pi$$

IF

$$\left|\int_{\gamma_2} f(x)dz\right| \to 0 \ as\ R \to \infty$$

$$\Rightarrow \int_{\gamma_1} f(z)dz = \int_{-R}^{R} dx \frac{1}{1+x^2} \to \pi(as\ R \to \infty)$$

$$R \to \infty, \left|\int_{\gamma_2} f(x)dz\right| \to 0$$

d)

Use

$$f(z) = \frac{1}{(1+z^2)^2}$$

On $\gamma = \gamma_1 + \gamma_2$ of part c

$$f(z) = \frac{1}{(z-i)^2(x+i)^2}$$

$f(z)$ has a double pole at $z = \pm i$

$$\int_{\gamma_1+\gamma_2} f(z)dz = 2\pi i a_{-1}$$

$a_{-1} \to z = i$

$$g(z) = (z-1)^2 f(z) \equiv \frac{1}{(z+i)^2}$$

Expand this about z=I

Note

$$f(z) = \frac{1}{(z+i)^2} \equiv \sum_{n=0}^{\infty} \frac{f^n(z)}{n!}\Big|_{z=i} (z-i)^n$$

Let $z = i + \Delta, (\Delta = z - i)$

$$\frac{1}{(z+1)^2} = \frac{1}{(2i+\Delta)^2} = \left(\frac{1}{2i}\right)^2 \frac{1}{\left(1+\frac{\Delta}{2i}\right)^2}$$

$$= -\frac{1}{4}\left(i - 2\frac{\Delta}{2i}\right) - -\frac{1}{4}(1 + i\Delta)$$

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On γ_2

$$|f(z)| = \left|\frac{1}{1+z^2}\right|^2 \leq \frac{1}{(r^2-1)^2}$$