Electromagnetism II

31 January 2012
09:59

Syllabus
- Maxwell's equations (Vacuum)<-Fundamental
- Electromagnetic waves
- Maxwell's equations (in media)<- useful approx. in some applications
- Electrostatics
- Magnetism
- Field theory-> radiation from a dipole

Comments:
- No new maths ಠ_ಠ
- Only new experiment
- We will use: Vector calculus
  - Electricity & magnetism (matter & fields II)

Books
- Introduction to electromagnetism- Griffiths
- Feynman lectures in physics

Vector Calculus- A Reminder

31 January 2012
10:19

Let us represent any position in this room by cartesian coordinates
If we measure the temperature at every point in the room, we get a scalar function T(x,y,z)
We can use the gradient operator to determine how this function changes

$$grad\ T = \underline{\nabla}T = \begin{pmatrix} \frac{\delta T}{\delta x} \\ \frac{\delta T}{\delta y} \\ \frac{\delta T}{\delta z} \end{pmatrix}$$

This is a vector quantity

$\underline{\nabla}T$ gives the direction in which the function is changing most quickly and how quickly it is changing

We have defined $\underline{\nabla} = \left(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}\right)$

$\underline{V} \times \underline{\nabla} \neq \underline{\nabla} \times \underline{V}$

We can use index notation

$(\underline{\nabla} \times \underline{V})_i = \varepsilon_{ijk}\delta_j V_k$ (*)

$$\varepsilon_{ijk} = \begin{cases} +1 & if\ \{i,j,k\} = \{1,2,3\},\{2,3,1\},\{3,1,2\} \\ -1 & if \{i,j,k\} = \{3,2,1\},\{2,1,3\},\{1,3,2\} \\ 0 & all\ others \end{cases}$$

In (*) we have a single index on both sides
(*) holds for each value of I
Ie i=1,i=2 and i=3
(*) is 3 scalar equations
J and k indices are both paired=> both are summed over
Exercise= Expand (*) and confirm it agrees with non-index way

Divergence->sinks & sources
$\underline{\nabla} * \underline{V} = \delta_i V_i$

Curl
$[\underline{\nabla} \times \underline{V}]_i = \varepsilon_{ijk}\delta_j V_k$

Curl⇔ Circulation
Divergence⇔ Sources/sinks

We also have the grad operation. Given a scalar field f, we can generate a vector $\underline{v}$ using

$\underline{v} = \underline{\nabla} f$

$[v_i = \delta_i f]$

We can't get a vector field this way
In particular, if $\underline{v} = \underline{\nabla} f$ then $\underline{\nabla} \times \underline{v} = 0$
Proof

$[\underline{\nabla} \times \underline{v}]_i = \varepsilon_{ijk}\delta_j v_k = \varepsilon_{ijk}\delta_j\delta_k f$

Recall

$$\frac{\delta}{\delta y}\frac{\delta}{\delta x} f = \frac{\delta}{\delta x}\frac{\delta}{\delta y} f\ etc$$

$$\Rightarrow \delta_j\delta_k f = \delta_k\delta_j f$$

$\delta_j\delta_k = \delta_k\delta_j$

ε_{ijk} =antisymmetric under $j \leftrightarrow k$

$[\underline{\nabla} \times \underline{v}]_i = \varepsilon_{ijk}\delta_j\delta_k f$

We can re-label pair indices $j \to a, k \to b$

$[\underline{\nabla} \times \underline{v}]_i = \varepsilon_{iab}\delta_a\delta_b f$

Re-label $a \to k, b \to j$

$[\underline{\nabla} \times \underline{v}]_i = \varepsilon_{ikj}\delta_k\delta_j f$

This relabeling effectively swaps $j \leftrightarrow k$

$[\underline{\nabla} \times \underline{v}]_i = \varepsilon_{ikj}\delta_k\delta_j f = -\varepsilon_{ijk}\delta_j\delta_k f = \varepsilon_{ijk}\delta_j\delta_k f = 0$

So $\underline{\nabla} \times \underline{\nabla}\mathrm{f} = -\underline{\nabla}\mathrm{x}\underline{\nabla}\mathrm{f} \Rightarrow \underline{\nabla} \times \underline{\nabla}\mathrm{f} = 0$

Divergence (Gauss) Theorem

Given a volume bounded by a surface S with an outward normal vector dS

Then

$$\int_V dV\, \underline{\nabla} * \underline{V} = \int_S \underline{v} * d\underline{S}$$

Stokes' Theorem

Given a closed path b with line element $d\underline{l}$ and any surface S bounded by b

$$\oint_b \underline{V} d\underline{l} = \int_S (\underline{\nabla} \times \underline{V}) * d\underline{S}$$

And Now for Something Completely Different

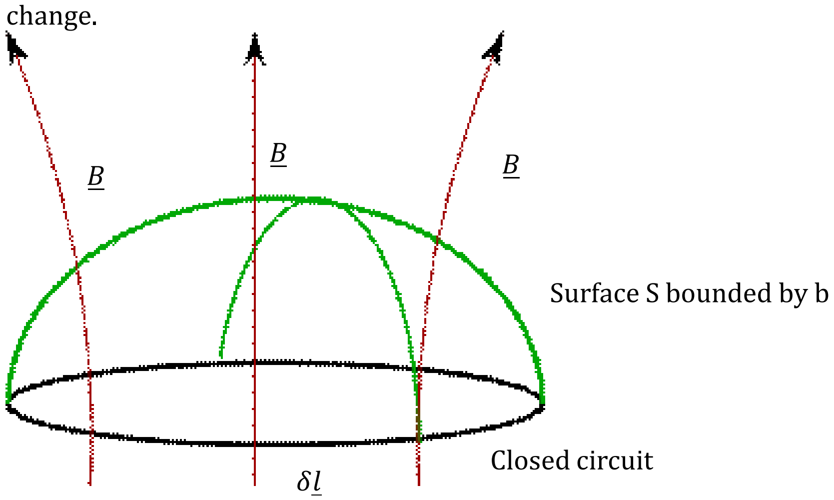
07 February 2012
10:30

1. Gauss' law

   Integral over any closed surface of $\underline{E} * d\underline{S} = \frac{1}{\epsilon_0}$ (charge enclosed)
   $d\underline{S}$ is outward normal to surface
   Let the charge density be $\rho(x)$
   Charge enclosed=
   $$= \int_V dV\ \rho$$
   Gauss' Law
   $$\rightarrow \int_S \underline{E} d\underline{S} = \frac{1}{\epsilon_0} \int_V dV\ \rho$$
   Divergence th'm
   $$= \int_V dV\ (\underline{\nabla} * \underline{E})$$
   $$\Rightarrow \int_V dV \left[\underline{\nabla} * \underline{E} - \frac{\rho}{\epsilon_0}\right] = 0$$
   This must hold for <u>any</u> volume ⇒ integral must vanish
   $$\Rightarrow \boxed{\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0}}$$
   [M1]
2. No magnetic monopoles

   No magnetic monopoles have been detected
   ⇒No sources/sinks for magnetic field
   $$\Rightarrow \boxed{\underline{\nabla} * \underline{B} = 0}$$
   [M2]
3. Faraday & Lenz's Law

   Whenever the magnetic flux linking a circuit changes, an EMF is induced in the circuit. The induced EMF has magnitude proportional to the rate of change of flux and its direction is such as to oppose the change.



   As there are no magnetic monopoles, there is nothing for the $\underline{B}$ to start/end on
   ⇒every line passing through b must pass through the surface S (for ANY surface bounded by b)
   We can use a surface integral to "count" the $\underline{B}$ lines
   The magnetic flux through the circuit
   $$\Phi = \int_S \underline{B} d\underline{S}$$
   The induced EMF is
   $$\int_b \underline{E} d\underline{l}$$

$$\int_b \underline{E} d\underline{l} = -\int_S \left(\frac{\delta \underline{B}}{\delta t}\right) d\underline{S}$$

Lenz's minus sign- EMF opposes change

Use Stoke's theorem

$$\int_b \underline{E} d\underline{l} = \int_S (\underline{\nabla} \times \underline{E}) d\underline{S}$$

Apply to faraday & Lenz

$$\int_S (\underline{\nabla} \times \underline{E}) * d\underline{S} = -\int_S \left(\frac{\delta \underline{B}}{\delta t}\right) * d\underline{S}$$

$$\Rightarrow \int_S \left(\frac{\delta \underline{B}}{\delta t} + \underline{\nabla} \times \underline{E}\right) * d\underline{S} = 0$$

This must hold for ANY surface ⇒integrand vanishes

$$\Rightarrow \frac{\delta \underline{B}}{\delta t} + \underline{\nabla} \times \underline{E} = 0 \Rightarrow \boxed{\underline{\nabla} \times \underline{E} = -\frac{\delta \underline{B}}{\delta t}}$$

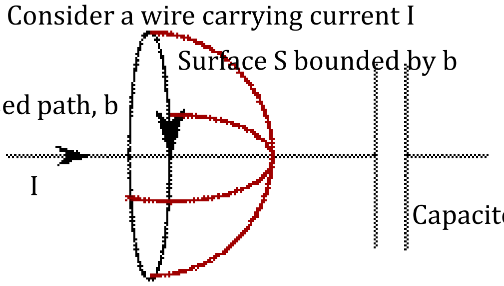
[M3]

4. Ampere's Law

Integral of $\underline{B} * d\underline{l}$ around any closed path $= \mu_0 x$ (current enclosed)

Unfortunately, this isn't quite correct

Consider a wire carrying current I



We can use the surface S to measure the current through the loop. If the current density is $\underline{j}$

$$I = \int_S \underline{j} d\underline{S}$$

Ampere would give

$$\int_b \underline{B} * d\underline{l} = \mu_0 \int_S \underline{j} d\underline{S}$$

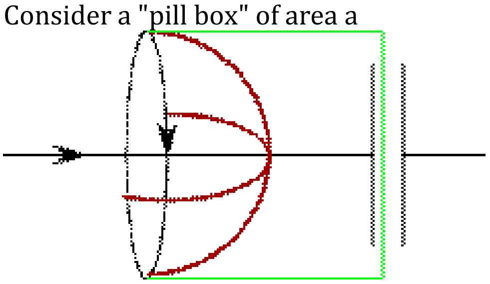
Problem: If our surface passes between the plates of a capacitor, there is no current.

Consider a parallel plate capacitor

If the total charge on each plate is $\pm Q$ we have charge densities $\pm\sigma = \pm\frac{Q}{A}$

To find the electric field, we use Gauss' law.

Consider a "pill box" of area a



Charge enclosed $= \sigma a$

Neglecting edge effects, $\underline{E}$ is perp to plates

$$\int_{surface\ of\ pillbox} \underline{E} d\underline{S} = |E| a$$

By Gauss' law

$$|E| = \frac{\sigma}{\epsilon_0}$$

Now think of our Ampere's law surface that cuts down the centre of the capacitor

$$\int_S \underline{E} d\underline{S} = \frac{\sigma}{\epsilon_0} A = \frac{Q}{\epsilon_0} = charge\ on\ plate$$

Taking time derivative, $\frac{\delta \underline{E}}{\delta t}$ over the surface, for parallel plate capacitor

$$\epsilon_0 \int_S \dot{\underline{E}} * d\underline{S} = I$$

(shown last time)***

⇒we modify ampere's law

$$\int_{closed\ path} \underline{B} d\underline{l} = \mu_0 \int_S d\underline{S} * \left(\underline{j} + \epsilon_0 \frac{\delta \underline{E}}{\delta t}\right)$$

Extra bit is "displacement current

Integral is now independent of the surface

Ampere's Law corrected

$$\int_b \underline{B} d\underline{l} = \mu_0 \int_S d\underline{S} * \left(\underline{j} + \epsilon_0 \frac{\delta \underline{E}}{\delta t}\right)$$

Now apply Stokes' theorem

$$\int_b \underline{B} d\underline{l} = \int_S (\underline{\nabla} \times \underline{B}) d\underline{S} = \mu_0 \int_S d\underline{S} * \left(\underline{j} + \epsilon_0 \frac{\delta \underline{E}}{\delta t}\right)$$

$$\Rightarrow \int_S d\underline{S} * \left(\mu_0 \left[\underline{j} + \epsilon_0 \frac{\delta \underline{E}}{\delta t}\right] - \underline{\nabla} \times \underline{B}\right) = 0$$

This holds for any surface ⇒ integrand vanishes

$$\Rightarrow \boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t}}$$

[M4]

Maxwell Equations

$$\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0}$$

[M1]

$$\underline{\nabla} * \underline{B} = 0$$

[M2]

$$\underline{\nabla} \times \underline{E} = -\frac{\delta \underline{B}}{\delta t}$$

[M3]

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t}$$

[M4]

In vacuum, $\rho = 0, \underline{j} = 0$

$$\underline{\nabla} * \underline{E} = 0$$

$$\underline{\nabla} * \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{\delta \underline{B}}{\delta t}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t}$$

Recap

We will need

$$(\underline{\nabla} \times (\underline{\nabla} \times \underline{v}))$$

$$[\underline{\nabla} \times (\underline{\nabla} \times \underline{v})]_i = \varepsilon_{ijk} \delta_j (\underline{\nabla} \times \underline{v})k = \varepsilon_{ijk} \delta_j \varepsilon_{klm} \delta_l v_m$$

$$= \varepsilon_{kji} \varepsilon_{klm} \delta_j \delta_l v_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \delta_j \delta_l = \delta_i (\delta_j v_j) - (\delta_j \delta_j) v_i$$

$$= \delta_i (\underline{\nabla} * \underline{v}) - (\nabla^2) v_i$$

$$\Rightarrow \boxed{\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \underline{\nabla}(\underline{\nabla} * \underline{v}) - \nabla^2 \underline{v}}$$

Vacuum solutions of Maxwell equations

Start with [M3]

$$\underline{\nabla} \times \underline{E} = -\frac{\delta \underline{B}}{\delta t}$$

Take curl of both sides

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla} \times \left(-\frac{\delta \underline{B}}{\delta t}\right)$$

$$\underline{\nabla}(\underline{\nabla} * \underline{E}) - \nabla^2 \underline{E} = -\frac{\delta}{\delta t} (\underline{\nabla} \times \underline{B})$$

$(\underline{\nabla} * \underline{E}) = 0$ because [M1]

$(\underline{\nabla} \times \underline{B}) \rightarrow use\ [M4]$

$$\Rightarrow -\nabla^2 \underline{E} = -\frac{\delta}{\delta t}\left(\mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t}\right)$$

$$\Rightarrow \left(\mu_0 \epsilon_0 \frac{\delta^2}{\delta t^2} - \nabla^2\right) \underline{E} = 0$$

Wave equation→ light in a vacuum?

Consider a plane wave solution $\underline{E} = \underline{E}_0 \cos(x - vt)$

$$\frac{\delta}{\delta t} \underline{E} = \underline{E}_0 v \sin(x - vt)$$

$$\frac{\delta^2}{\delta t^2} \underline{E} = -\underline{E}_0 v^2 \cos(x - vt)$$

$$\nabla^2 \underline{E} = \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right) \underline{E}$$

$$\frac{\delta \underline{E}}{\delta x} = -\underline{E}_0 \sin(x - vt)$$

$$\frac{\delta^2 \underline{E}}{\delta x^2} = -\underline{E}_0 \cos(x - vt)$$

Sub into wave equation

$$-(\mu_0 \epsilon_0 v^2 - 1)\underline{E}_0 \cos(x - vt) = 0$$

Of this is to hold for all x and t, we need

$$v^2 = \frac{1}{\mu_0 \epsilon_0} \Rightarrow \boxed{v = c}$$

Our solution

$$\underline{E} = \underline{E}_0 \cos(x - ct)$$

Satisfies the wave equation. What about the other maxwell equations?

[M1]

$$\underline{\nabla} * \underline{E} = 0$$

(we're in a vacuum)

$$\Rightarrow \frac{\delta}{\delta x} E_x + \frac{\delta}{\delta y} E_y + \frac{\delta}{\delta z} E_z = 0$$

$$\Rightarrow \frac{\delta}{\delta x} E_x = 0$$

$$\Rightarrow (E_0)_x (-\sin(x - ct)) = 0$$

We need this to hold for all x and t

$\Rightarrow (E_0)_x = 0 \Rightarrow \underline{E}$ is perpendicular to direction of travel

We can arrange our axes so that

$$\underline{E} = \begin{pmatrix} 0 \\ E_0 \cos(x - ct) \\ 0 \end{pmatrix}$$

Wave eqn✓

$\underline{\nabla} * \underline{E} = 0$✓

Now use

$$\underline{\nabla} \times \underline{\mathrm{E}} = -\frac{\delta \underline{\mathrm{B}}}{\delta \mathrm{t}}$$

To find $\underline{B}$

$$\Rightarrow \frac{\delta \underline{B}}{\delta t} = -\underline{\nabla} \times \underline{E} = -\begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 0 & E_0 \cos(x - ct) & 0 \end{vmatrix} = -\begin{pmatrix} 0 \\ 0 \\ -E_0 \sin(x - ct) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ E_0 \sin(x - ct) \end{pmatrix}$$

Integrating with respect to time

$$\underline{B} = \begin{pmatrix} 0 \\ 0 \\ \frac{E_0}{c} \cos(x - ct) \end{pmatrix} + \text{integration "constant"}$$

Integration "constant" has no time dependence

→ not relevant for the EM radiation

Handwavium FTW

$$\underline{E} = \begin{pmatrix} 0 \\ E_0 \cos(x - ct) \\ 0 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} 0 \\ 0 \\ \frac{E_0}{c}\cos(x - ct) \end{pmatrix}$$

Wave eqn✓

$\underline{\nabla} * \underline{E} = 0$✓

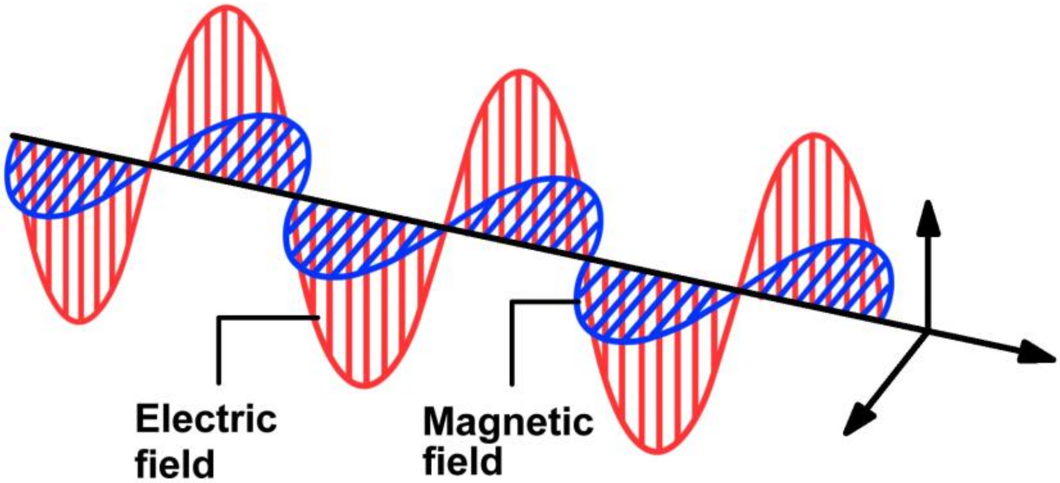
$\underline{\nabla} * \underline{B} = 0$✓

$\underline{\nabla} \times \underline{E} = -\frac{\delta \underline{B}}{\delta t}$✓

$\underline{\nabla} \times \underline{B} = \mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t}$

We have a wave moving at speed c in the x- direction

$\underline{E}$ and $\underline{\mathrm{B}}$ are mutually orthogonal and orthogonal to the direction of travel



And god said:

$$\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0} \qquad \underline{\nabla} * \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{\delta \underline{B}}{\delta t} \quad \underline{\nabla} \times \underline{\mathrm{B}} = \mu_0 \underline{\mathrm{j}} + \mu_0 \epsilon_0 \frac{\delta \underline{\mathrm{E}}}{\delta \mathrm{t}}$$

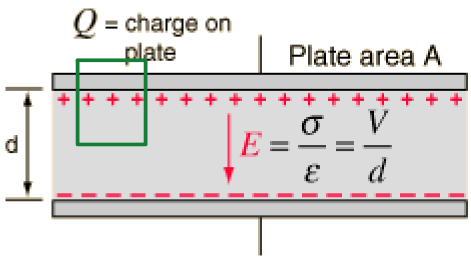
And THEN there was light

Back to the Basics

09 February 2012

12:33

For $\underline{E}$ fields, consider a parallel plate capacitor



To find $\underline{E}$, we use a Gauss pillbox of area a

Gauss' law

$$\int_S \underline{E}\, d\underline{S} = \frac{1}{\epsilon_0} \times (\text{charge enclosed})$$

If the charge per unit area is σ

$$|E|a = \frac{\sigma a}{\epsilon_0} \Rightarrow |E| = \frac{\sigma}{\epsilon_0}$$

If the plate spacing is d, the potential difference across the plates is $V = |E|d = \frac{\sigma d}{\epsilon_0}$

If the plates have area L^2, the total charge on the plate $Q = \sigma L^2$

And we have

$$V = \frac{Qd}{L^2 \epsilon_0}$$

The work done in moving a charge dQ against a voltage V is VdQ

The work done in charging the capacitor is

$$WD = \int_0^{Q_F} V dQ = \int_0^{Q_F} \frac{Qd}{L^2 \epsilon_0} dQ = \frac{d}{L^2 \epsilon_0} \left[\frac{Q^2}{2}\right]_0^{Q_F} = \frac{dQ_F^2}{2L^2 \epsilon_0}$$

This energy is stored in the $\underline{E}$ field

Using

$$|E| = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 L^2}$$

We have $Q = \epsilon_0 L^2 |E|$

$$WD = \frac{1}{2} \frac{d}{L^2 \epsilon_0} (\epsilon_0 L^2 |E|)^2 = \frac{1}{2} dL^2 \epsilon_0 |E|^2$$

Now dL^2 is the volume in which the electric field exists

Energy stored in electric field=WD in charging capacitor

$$= \frac{\epsilon_0}{2} |E|^2 * (\text{volume})$$

$$\Rightarrow \boxed{\text{energy density in electric field} = \frac{\epsilon_0}{2} |E|^2}$$

By studying a capacitor, we find $\boxed{\text{energy density in electric field} = \frac{\epsilon_0}{2} |E|^2}$

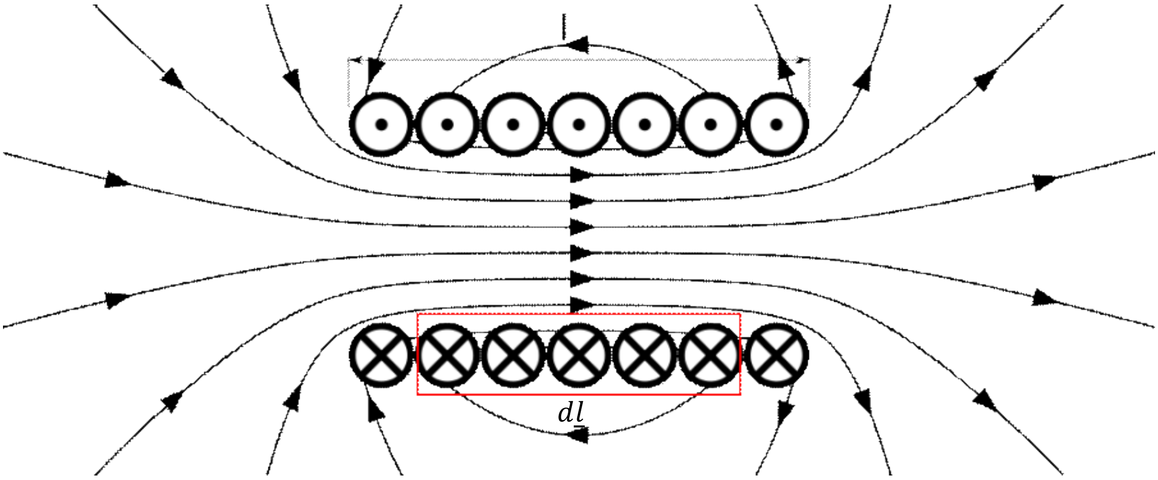
<u>Magnetic field energy density</u>

For $\underline{B}$ fields, we can consider a solenoid.

Let solenoid have n turns per unit length and carry a current I

Solenoid is long& straight (length L) and area A

In cross section



$\oint \underline{B} d\underline{l} = \mu_0 x(\text{current enclosed})$

For the ampere circuit drawn,

$\oint \underline{B} d\underline{l} = |\underline{B}|S$

The current enclosed=nSI

Ampere's law

$\Rightarrow |B|S = nSI\mu_0 \Rightarrow |B| = \mu_0 nI$ (Uniform throughout interior of solenoid)

The magnetic flux per turn of the coil is $|B|A = \mu_0 nIA$

The solenoid has length $L \Rightarrow nL$ turns in total

$\Rightarrow$ total flux through solenoid is $n^2\mu_0 LAI$

If I changes, by faraday+lenz's law we have an $EMF = -n^2\mu_0 LA\dot{I}$

The work done in "energising" the solenoid is

$$WD = \int VdQ = \int VIdt = n^2\mu_0 LA \int \dot{I}Idt = n^2\mu_0 LA\frac{I^2}{2}$$

$$\Rightarrow WD = \frac{LA}{2\mu_0}(nI\mu_0)^2 = \frac{LA}{2\mu_0}\underline{B}\underline{B}$$

The work done is stored in the magnetic field

$\Rightarrow$ energy in magnetic field $= LA\frac{\underline{B}\underline{B}}{2\mu_0}$

LA=volume of solenoid$\Rightarrow$ energy density in $\underline{B}$ field=$\frac{\underline{B}\underline{B}}{2\mu_0}$

In general, for $\underline{E}$ and $\underline{B}$ fields, total energy stored in fields is

$$\text{Energy in fields} = \int dV\left(\frac{\epsilon_0}{2}\underline{E}\underline{E} - \frac{\underline{B}\underline{B}}{2\mu_0}\right)$$

Continuity equations

(The change in amount of stuff in a box)=(stuff put in)-(stuff taken out)

e.g. bank account

(Rate The change in amount of stuff in a box)=(Rate stuff put in)-(Rate stuff taken out)

Apply this to the energy in the $\underline{E}$&$\underline{B}$ fields in some fixed volume V

IF the total energy is

$$\epsilon = \int dV \left(\frac{\epsilon_0}{2}\underline{E}\underline{E} - \frac{\underline{B}\underline{B}}{2\mu_0}\right)$$

$$\Rightarrow \frac{\delta\epsilon}{\delta t} = \int dV \left(\epsilon_0\underline{E}\underline{\dot{E}} - \frac{\underline{B}\underline{\dot{B}}}{\mu_0}\right)$$

Recall in vacuum

$\underline{\nabla} \times \underline{E} = -\underline{\dot{B}}$

$\underline{\nabla} \times \underline{B} = \mu_0\epsilon_0\underline{\dot{E}}$

$$= \frac{1}{\mu_0}\int dV \left(\underline{E}(\underline{\nabla} \times \underline{B}) - \underline{B}(\underline{\nabla} \times \underline{E})\right)$$

What is this?

Consider

$\underline{\nabla}(\underline{E} \times \underline{B}) = \delta_i(\underline{E} \times \underline{B})_i = \delta_i[\epsilon_{ijk}E_jB_k]$

$= \epsilon_{ijk}\delta_i(E_jB_k)$

ϵ_{ijk} is a constant

$= \epsilon_{ijk}(B_k(\delta_iE_j) + E_j(\delta_iB_k))$

Product rule

$= B_k \epsilon_{kij} (\delta_i E_j) - E_j \epsilon_{jik} \delta_i B_k$

ϵ_{kij} →Cyclic order preserved in ϵ_{ijk}

ϵ_{jik} →1 pair of indices swapped → -1 factor

$= B_k (\underline{\nabla} \times \underline{E})_k - E_j (\underline{\nabla} \times \underline{B})_j = \boxed{\underline{B} * (\underline{\nabla} \times \underline{E}) - \underline{E} * (\underline{\nabla} \times \underline{B}) = \underline{\nabla}(\underline{E} \times \underline{B})}$ ← starting point

$$\frac{\delta\epsilon}{\delta t} = \frac{1}{\mu_0} \int dV \left(-\underline{\nabla}(\underline{E} \times \underline{B})\right) = \frac{-1}{\mu_0} \int_S d\underline{S}(\underline{E} \times \underline{B})$$

Surface S bounds volume V →divergence theorem

$\frac{-1}{\mu_0} \int_S d\underline{S}(\underline{E} \times \underline{B})$ ⇒Energy flux

$$\frac{\delta\epsilon}{\delta t} = -\int_S d\underline{S} \frac{1}{\mu_0} (\underline{E} \times \underline{B})$$

Often, we can identify

$$\frac{(\underline{E} \times \underline{B})}{\mu_0}$$

With energy flux out of the box, but beware! e.g. consider a solenoid inside a capacitor which gives static crossed $\underline{E}$ & $\underline{B}$ fields

$\frac{(\underline{E} \times \underline{B})}{\mu_0}$ is called the Poynting vector

Energy Density & Flux in E.M. Waves

Recall simple EM wave

$$\underline{E} = \begin{pmatrix} 0 \\ E_0 \cos(x - ct) \\ 0 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} 0 \\ 0 \\ \frac{E_0}{c} \cos(x - ct) \end{pmatrix}$$

Energy Density:

$$\frac{\varepsilon_0}{2} \underline{E}\underline{E} + \frac{\underline{B}\underline{B}}{2\mu_0} = \frac{\varepsilon_0}{2} E_0^2 \cos^2(x - ct) + \frac{E_0^2}{2c^2\mu_0} \cos^2(x - ct)$$

$\underline{E}$ & $\underline{B}$ fluctuate → we average over a cycle

$$< \cos^2(x - ct) > \ = \frac{1}{2}$$

$$\Rightarrow \text{<Energy Density>} = \frac{\varepsilon_0 E_0^2}{4} + \frac{E_0^2}{4c^2\mu_0}$$

$$\left(\text{Using } c^2 = \frac{1}{\varepsilon_0 \mu_0}\right)$$

$$= \frac{\varepsilon_0 E_0^2}{4} + \frac{\varepsilon_0 E_0^2}{4} = \frac{\varepsilon_0 E_0^2}{2}$$

Energy Flux

(can use Poynting vector here)

$$< \frac{1}{\mu_0} \underline{E} \times \underline{E} > \ = \ < \frac{1}{\mu_0} \frac{E_0^2}{c} \cos^2(x - ct) > \underline{\hat{x}}$$

$\underline{\hat{x}}$ =unit vector in x direction

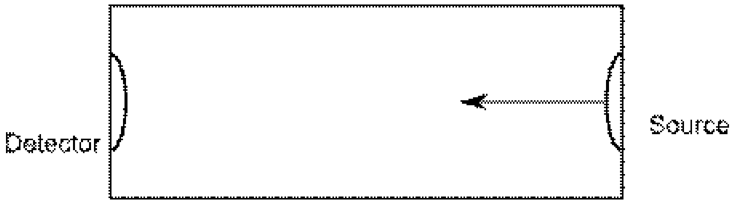
$$= \frac{1}{2} \frac{E_0^2}{\mu_0 c} \underline{\hat{x}} = c \left(\frac{\varepsilon_0 E_0^2}{2}\right) \underline{\hat{x}}$$

$$< \text{Energy Flux} > \ = c < \text{Energy Density} > \underline{\hat{x}}$$

Momentum

Einstein argument: Consider a spaceship in deep space (no external forces)

A light source on one side of the ship sends a flash of light across to a detector on the other side



Let the light pulse have total energy E
The source battery converts a tiny mass $m = E/c^2$ into energy to produce the light
At the far end, the detector converts the light back to chemical energy in its battery
$\Rightarrow$ detector battery gets heavier by $m = E/c^2$
If the ship doesn't move, the centre of mass of the system moves with no external force acting

<u>Resolution</u>

The light carries momentum p in the direction of travel
When the light is emitted, the ship recoils with speed $v_{recoil} = P/M$
M=mass of ship
If the detector is distance L from the source, the light travel time is just L/c
In this time, the ship moves a distance $\frac{P}{m}\frac{L}{c}$
When the light is absorbed, it transfers its momentum back to the ship=>ship stops
To keep centre of mass fixed,

$$\underset{\leftarrow}{\text{Mass} * \text{distance moved}} = \underset{\rightarrow}{\text{Mass}*\text{distance moved}}$$

$$mL = M\frac{P}{M}\frac{L}{c}$$

$$m = \frac{P}{c}$$

Then the momentum carried by the light
$= c \times$ (mass converted to produce the light)

$$= c * \frac{E}{c^2} \Rightarrow \boxed{P = \frac{E}{c}}$$

For our simple wave
Energy density, u, $< u > \; = \frac{1}{2}\varepsilon_0 E_0^2$
Average energy flux $c < u > \hat{\underline{x}}$
Average momentum density $< \underline{P} > \; = \frac{<u>}{c}\hat{\underline{x}}$

Media 1: Simple

15 February 2012
11:49

"Stuff has got bits in it"- in particular, nuclei + electrons⇒ charged particles that ca act as sources of currents

<u>Option 1</u>: remember the bits and use

$$\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0} \qquad \underline{\nabla} * \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{\delta \underline{B}}{\delta t} \quad \underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\delta \underline{E}}{\delta t}$$

<u>Problem</u>

There are many charges to worry about

<u>Option 2</u>: incorporate bulk effects of all the charges into modifications of the maxwell eqns then ignore the presence of the charges

This is an approximation/fudge

<u>Media 1: Conductors</u>

<u>Plan</u>

Incorporate the bulk properties of the medium into our Maxwell's equations rather than work at the individual electron level. → we will be making approximations

We will consider "simple" materials

Simple means

1. Linear →Response is proportional to stimulus
2. Isotropic →No preferred directions in the material ⇒ response is proportional to the stimulus

<u>Example</u>

In a simple conductor the current density $\underline{j}$ is related to the applied field by

$$\underline{j} = \sigma \underline{E}$$

σ is conductivity

Now consider maxwell's equations in a conducting medium

We have $\rho = 0, \underline{j} = \sigma \underline{E}$

M1 becomes

$$\underline{\nabla}.\underline{E} = 0, \qquad \underline{\nabla} \times \underline{E} = -\dot{\underline{B}}$$

$$\underline{\nabla}.\underline{B} = 0, \qquad \underline{\nabla} \times \underline{B} = \mu_0 (\underline{j} + \epsilon_0 \dot{\underline{E}})$$

$$= \mu_0 (\sigma \underline{E} + \epsilon_0 \dot{\underline{E}})$$

We use the same game as we did in vacuum, i.e.

Take curl of $\underline{\nabla} \times \underline{E} = -\dot{\underline{B}}$

$$\Rightarrow \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\underline{\nabla} \times (\dot{\underline{B}})$$

$$\Rightarrow \underline{\nabla}(\underline{\nabla}.\underline{E}) - \nabla^2 \underline{E} = -\frac{\delta}{\delta t}(\underline{\nabla} \times \underline{B})$$

$\underline{\nabla}.\underline{E} = 0$

By m1

$$\Rightarrow -\nabla^2 \underline{E} = -\frac{\delta}{\delta t}(\mu_0 \sigma \underline{E} + \mu_0 \epsilon_0 \dot{\underline{E}})$$

$$\Rightarrow \mu_0 \epsilon_0 \ddot{\underline{E}} - \nabla^2 \underline{E} + \mu_0 \sigma \dot{\underline{E}} = 0$$

(*)

$\mu_0 \sigma \dot{\underline{E}}$= additional term due to conductivity

By analogy with SHM, the additional term introduces damping

We guess the form of the solution

$$\underline{E} = \underline{E}_0 \exp[(-\lambda + ik)\hat{\underline{k}}\underline{x} - i\omega t]$$

$\hat{\underline{k}}$ unit vector in direction of propagation

For k and ω both +ve, the wave travels in the $\hat{\underline{k}}$ direction

We take our guess and substitute into (*)

$$\dot{\underline{E}} = -i\omega \underline{E}, \qquad \ddot{\underline{E}} = (-i\omega)^2 \underline{E} = -\omega^2 \underline{E}$$

As

$$\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$$

We also need the spatial derivatives

Remember

$$\underline{\hat{k}}.\underline{x} = \hat{k}_x x + \hat{k}_y y + \hat{k}_z z$$

$$\frac{\delta}{\delta x}\underline{E} = (-\lambda + ik)\hat{k}_x \underline{E}, \qquad \frac{\delta^2 \underline{E}}{\delta x^2} = (-\lambda + ik)^2 \hat{k}_x \underline{E}$$

$$\Rightarrow \nabla^2 \underline{E} = (-\lambda + ik)^2 \left(\hat{k}_x^2 + \hat{k}_y^2 + \hat{k}_z^2\right)\underline{E}$$

$$\underline{\hat{k}}.\underline{\hat{k}} = 1$$

Plugging into (*)

$$\mu_0 \epsilon_0 (-\omega^2)\underline{E} - (-\lambda + ik)^2 \underline{E} + \mu_0 \sigma(-i\omega)\underline{E} = 0$$

$$\Rightarrow (-\omega^2 \mu_0 \epsilon_0 - (-\lambda + ik)^2 - i\omega\mu_0\sigma)\underline{E} = 0$$

If this is to hold for all t and $\underline{x}$, we need

$$-\omega^2 \mu_0 \epsilon_0 - (-\lambda + ik)^2 - i\omega\mu_0\sigma = 0$$

Taking real and imaginary parts

$\mu_0 \epsilon_0 \omega^2 + \lambda^2 - k^2 = 0$

(real part)

$-2\lambda k + \mu_0 \sigma \omega = 0$

We have two constants ⇒ we can look for k and λ as a function of ω

If we pick the frequency of light we shine onto a conductor

The wavelength and damping are fixed

From the Im part,

$$k = \frac{\mu_0 \sigma \omega}{2\lambda}$$

Eliminate k from real part

$$\mu_0 \epsilon_0 \omega^2 + \lambda^2 = \left(\frac{\mu_0 \sigma \omega}{2\lambda}\right)^2$$

Multiplying by $4\lambda^2$ ⇒

$$4\lambda^4 + 4\lambda^2(\mu_0 \epsilon_0 \omega^2) - (\mu_0 \sigma \omega)^2 = 0$$

Quadratic in λ^2

$$\Rightarrow \lambda^2 = \frac{-4(\mu_0 \epsilon_0 \omega^2) \pm \sqrt{16(\mu_0 \epsilon_0 \omega^2)^2 + 16(\mu_0 \epsilon_0 \omega)^2}}{8}$$

We take $+\sqrt{\square}$ sol'n to ensure $\lambda^2 > 0$

For σ large, $\lambda^2 \to \frac{\sqrt{16(\mu_0 \sigma \omega)^2}}{8}$

$$\Rightarrow \lambda^2 = \frac{\mu_0 \sigma \omega}{2}$$

$$\Rightarrow \lambda = \sqrt{\frac{\mu_0 \sigma \omega}{2}}, \qquad k = \sqrt{\frac{\mu_0 \sigma \omega}{2}}$$

For LARGE σ

Frequency dependent dampening inside conductor

Higher frequency ⇒ faster attenuation

Field is only non-vanishing in the "skin" of the material

Recap

21 February 2012
10:07

Media 1

Conducting Media

Simple ≡ linear, isotropic
In a simple conductor

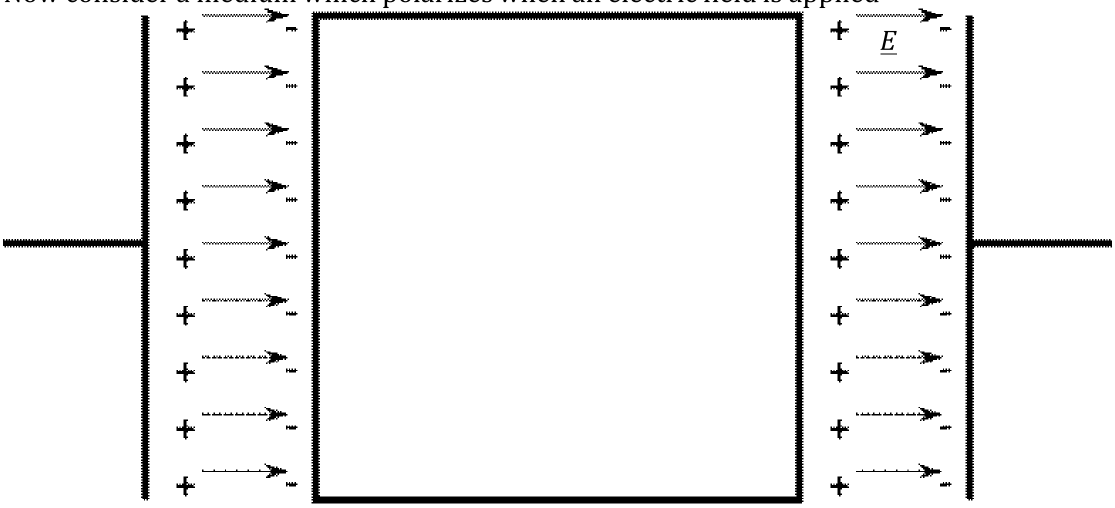
$\underline{j} = \sigma \underline{E}$

σ= conductivity
⇒modified wave equation
⇒ attenuation

Media 2: Dielectrics

21 February 2012
10:09

Now consider a medium which polarizes when an electric field is applied



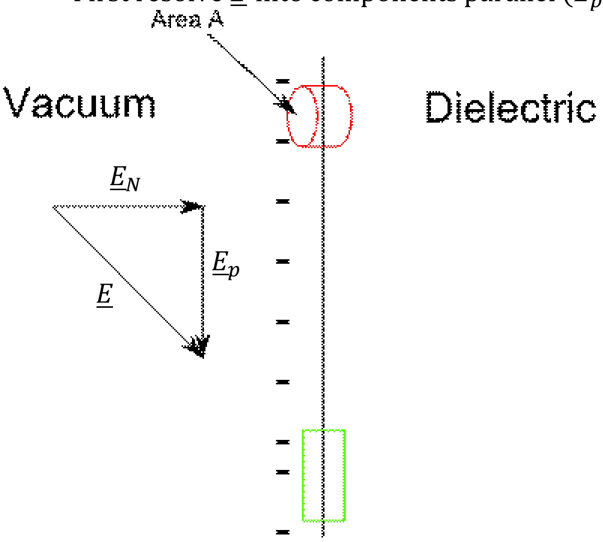
As unlike charges attract, the applied electric field will ionize SURFACE charges
BULK of material remains neutral

The effect of the surface charges is to <u>reduce</u> the component of the electric field <u>perpendicular</u> to the surface

<u>More general case</u>

Consider an electric field at some angle to the surface of dielectric.

First resolve $\underline{E}$ into components parallel (E_p) and normal (E_n) to surface



The normal component of the electric induces a surface charge on the dielectric

Drawing short fat Gaussian pillbox, we see that the surface charge only affects the normal component of $\underline{E}$

$$\int_S \underline{E}\,d\underline{S} = -E_{out}^N \times A + E_{in}^N \times A = \frac{1}{\epsilon_0} \times (\text{Charge Enclosed}) = \frac{A}{\epsilon_0} \times (\text{induced surface charge density})$$

$-E_{out}^N \times A$ Flat face in vacuum

$E_{in}^N \times A$ Flat face in dielectric

For a <u>simple</u>(linear response), the induced surface charge density is proportional to $E^N \Rightarrow E_{in}^N$ is some fraction of E_{out}^N

We can write

$$E_{in}^N = \frac{E_{out}^N}{\epsilon_r}$$

ϵ_r is the "relative permittivity" of the medium- material dependent constant

$\epsilon_r > 1$

<u>"Walking" an electron</u>

Consider taking a charge (e.g. electron) for a walk around the tall thin (green) path (anticlockwise) as shown. As the charge moves along the long sections of the path it experiences a force due to $\underline{E}$, the

component of the force parallel to the path is determined by $E_{in}^{p}(E_{out}^{p})$ inside (outside) the dielectric
The total work done in moving the charge around the complete circuit must be zero ⇒ work done on the charge by E_{out}^{p} while on the outside is recovered on the inside by E_{in}^{p} ⇒ $\boxed{E_{in}^{p} = E_{out}^{p}}$

That was the physics of dielectrics

Textbooks unfortunately then confuse matters by defining a range of new quantities (NOT EXAMINABLE- "an unfortunate historic aberration"- WBP)

The first is sensible: Polarization

$\underline{P} = \chi\epsilon_0\underline{E}$

χ= susceptibility

$[\epsilon_r = 1 + \chi]$

The second is there to confuse: "electric displacement" $\underline{D} = \epsilon_0\underline{E} + \underline{P}$

For simple dielectrics $\underline{D} = \epsilon_0\underline{E}$

Using

$$E_{in}^{N} = \frac{E_{out}^{N}}{\epsilon_r} \Rightarrow \epsilon_r E_{in}^{N} = E_{out}^{N} \Rightarrow D_{in}^{N} = D_{out}^{N}$$

~~

Maxwell in a dielectric

Rule:

Send $\epsilon_0 \to \epsilon_r\epsilon_0$ and forget dielectric ϵ_0 appears in Gauss' law and Ampere;s law

1. Consider gauss in a dielectric

Gauss

$$\int_S \underline{E}d\underline{S} = 1/\epsilon_0$$

(charge enclosed)

~

A charge +Q pulls electrons towards it producing dipoles as shown

Now consider a gaussian surface- the ends of some dipoles will be inside whilse their +ve ends are outside

⇒ polarisation of the medium reduces the total charge inside the Gauss surface.

Applying gauss, we find a reduced electric field produced by a factor of ϵ_r *This what really happens*

We want to take this by modifying Gauss' law

If forget the medium and use

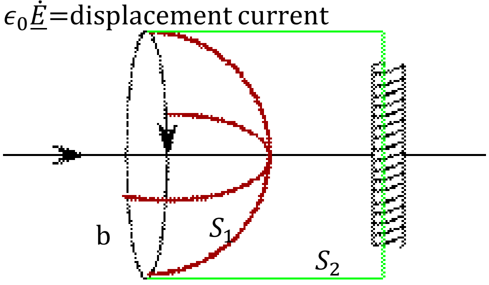
$$\int_S \underline{E}d\underline{S} = \frac{1}{\epsilon_r\epsilon_0}$$

We get the same answer as doing it properly i.e. sending $\epsilon_0 \to \epsilon_r\epsilon_0$ "fakes" it for Gauss' law

2. Dielectric Ampere's law

$$\oint_b \underline{B}d\underline{l} = \mu_0 \int_S (\underline{j} + \epsilon_0\dot{\underline{E}})d\underline{S}$$

$\epsilon_0\dot{\underline{E}}$=displacement current



We introduced the displacement current s that the right hand side of ampere's law gave the same answer for surface S_1 and surface S_2. Now consider filling the capacitor with a dielectric ⇒ $\underline{E}$ inside capacitor reduces electric field by a factor ϵ_r while the changing polarisation of the medium generates a current across S_2

Taking both effects into account, we get the same answer as before and $\int_{S_2}$ ⬚ still matches $\int_{S_1}$ ⬚

Again, we want to fake this

Using

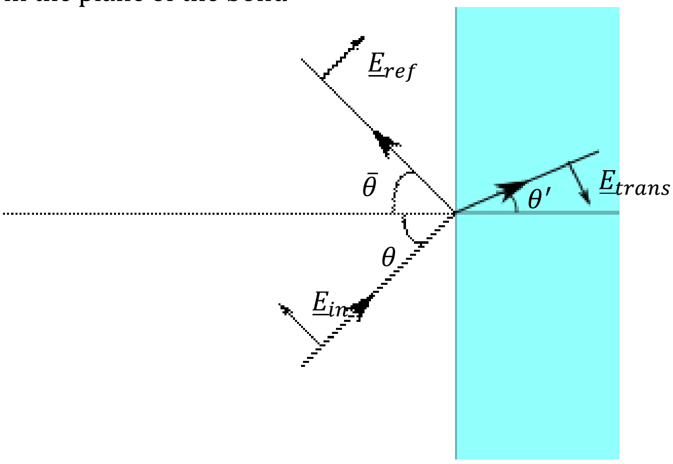
$$\oint \underline{B}d\underline{l} = \mu_0 \int_S (\underline{j} + \epsilon_r\epsilon_0\dot{\underline{E}})d\underline{S}$$

Allows us to forget the dielectric

Again, we send $\epsilon_0 \to \epsilon_r\epsilon_0$ and forget the dielectric

3. Waves in dielectrics

A constant is a constant ⇒ we get a simple wave equation again
The wave space in a vacuum, $c = 1/\sqrt{\epsilon_0\mu_0}$ ⇒ wave speed in the dielecric is $c' = 1/\sqrt{\epsilon_r\epsilon_0\mu_0} \equiv c/\sqrt{\epsilon_r}$
In optics the refractive index $n = \frac{c}{c'} = \sqrt{\epsilon_r}$
Consider light shining on to a slab of dielectric (e.g. glass) with polarisations as shown - all $\underline{E}$ fields are in the plane of the bond



$$\underline{E}_{in} = \underline{E}_{in}^0 \exp\left(-i\omega t + \frac{i\omega}{c}(\cos\theta x + \sin\theta y)\right)$$

$$\underline{E}_{ref} = \underline{E}_{ref}^0 \exp\left(-i\bar{\omega} t + \frac{i\bar{\omega}}{c}(-\cos\bar{\theta} x + \sin\bar{\theta} y)\right)$$

$$\underline{E}_{trans} = \underline{E}_{trans}^0 \exp\left(-i\omega' t + \frac{i\omega'}{c'}(\cos\theta' x + \sin\theta' y)\right)$$

We know that $E_{in}^p = E_{out}^p$ i.e. parallel component of $\underline{E}$ is continuous across the boundary

$$\Rightarrow |\underline{E}_{in}|\cos\theta + |\underline{E}_{ref}|\cos\bar{\theta}\Big|_{x=0^-} = |\underline{E}_{trans}|\cos\theta'\Big|_{x=0^+}$$

$x = 0^-$=just below 0
$x = 0^+$=just below 0

Substituting for the 3 waves

$$\Rightarrow |\underline{E}_{in}^0|\cos\theta \exp\left(-i\omega t + \frac{i\omega}{c}\sin\theta y\right) + |\underline{E}_{trans}^0|\cos\bar{\theta}\exp\left(-i\bar{\omega} t + \frac{i\bar{\omega}}{c}\sin\theta y\right)$$
$$= |\underline{E}_{ref}^0|\cos\theta' \exp\left(-i\omega' t + \frac{i\omega'}{c'}\sin\theta' y\right)$$

This must hold for all t and all y
At y=0, this takes the form

$$(\text{const})e^{-i\omega t} + (\overline{\text{const}})e^{-i\bar{\omega} t} = (\text{const}')e^{-i\omega' t}$$

For this to hold for all values of t, we need $\omega = \bar{\omega} = \omega'$
Now consider $y \neq 0$ and cancel common factor of $e^{-i\omega t}$,

$$(\text{const})\exp\left[\frac{i\omega}{c}\sin\theta y\right] + (\overline{\text{const}})\exp\left[\frac{i\omega}{c}\sin\bar{\theta} y\right] = (\text{const}')\exp\left[\frac{i\omega}{c}\sin\theta' y\right]$$

For this to hold for all y we need

$$\frac{\sin\theta}{c} = \frac{\sin\bar{\theta}}{c} = \frac{\sin\theta'}{c'}$$
$$\Rightarrow \theta = \bar{\theta}$$
$$\frac{c}{c'} = \frac{\sin\theta}{\sin\theta'} = \sqrt{\epsilon_r} = n$$

Snell's law

Imposing parallel component continuous gave

$$|\underline{E}_{in}^0|\cos\theta + |\underline{E}_{ref}^0|\cos\bar{\theta} = |\underline{E}_{trans}^0|\cos\theta'$$

We also have

$$E_N^{in} = \frac{E_N^{out}}{\epsilon_r} \Rightarrow E_N^{out} = \epsilon_r E_N^{in}$$
$$\Rightarrow -|\underline{E}_{in}^0|\sin\theta + |\underline{E}_{ref}^0|\sin\bar{\theta} = -\epsilon_r|\underline{E}_{trans}^0|\sin\theta'$$

Imposing $\theta = \bar{\theta}$ we need

$$\left(|\underline{E}_{in}^0| + |\underline{E}_{ref}^0|\right)\cos\theta = |\underline{E}_{trans}^0|\cos\theta'$$
$$\left(-|\underline{E}_{in}^0| + |\underline{E}_{ref}^0|\right)\sin\theta = \epsilon_r|\underline{E}_{trans}^0|\cos\theta'$$

As θ' is fixed by θ using snell's law, we have a pair of simultaneous equations for $|\underline{E}_{ref}^0|$ and $|\underline{E}_{trans}^0|$ as

functions of $|\underline{E}_{in}^0|$ and θ
Exercise

Brewster angle: show that

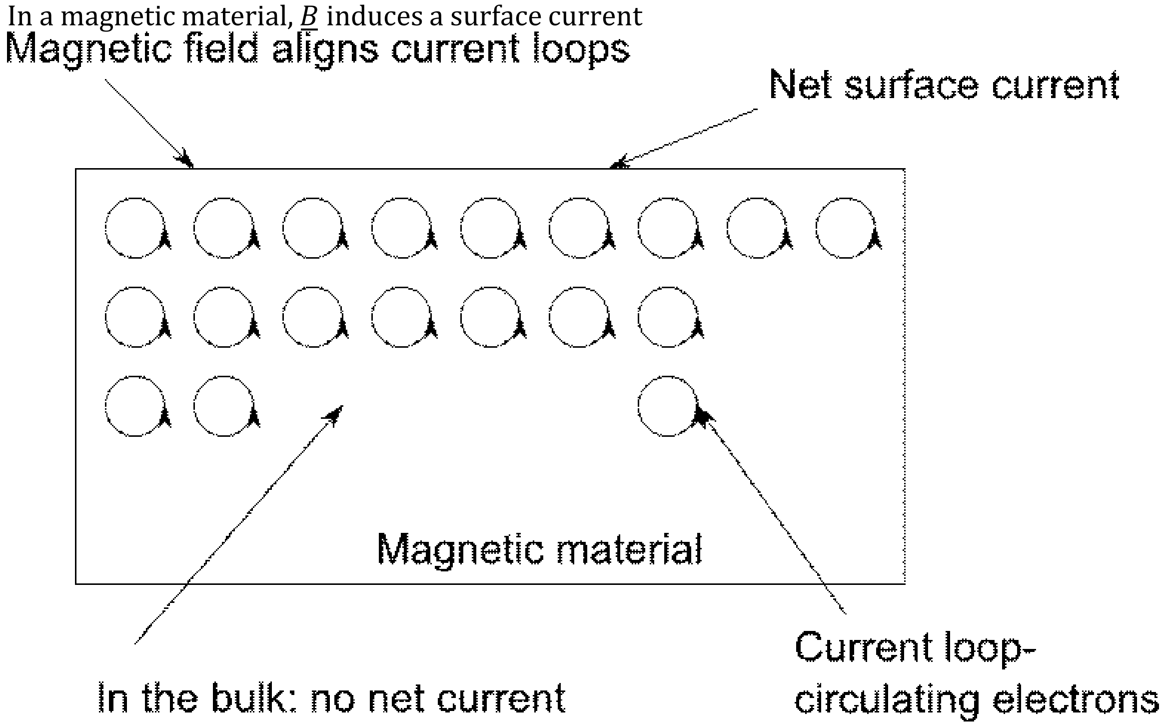
$$|\underline{E}_{ref}^0| = 0 \leftrightarrow \theta + \theta' = \frac{\pi}{2}$$

The other polarisation: you can do all of this for $\underline{E}$ perp to the board

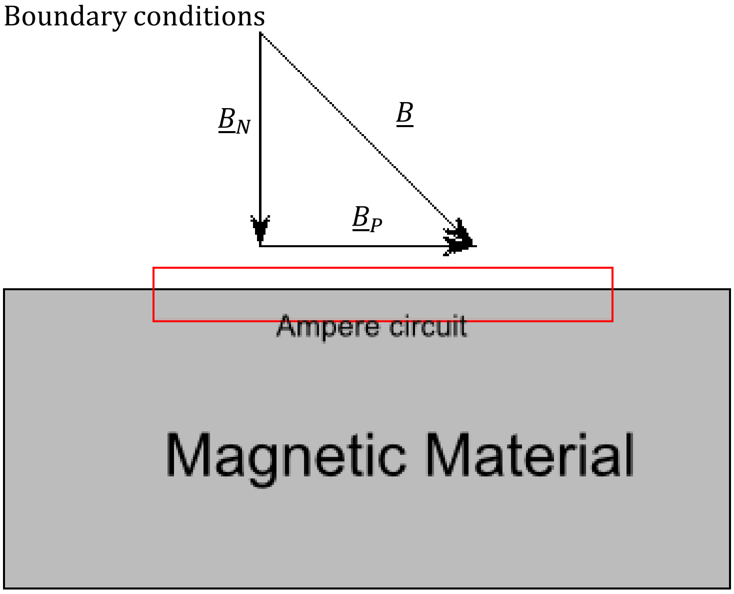
Media 3: Magnetic

22 February 2012
12:34

In a dielectric, $\underline{E}$ induces a surface charge
In a magnetic material, $\underline{B}$ induces a surface current



Boundary conditions



As there are no magnetic monopoles, the magnetic field lines can't end

$$\Rightarrow \boxed{B_N^{in} = B_N^{out}}$$

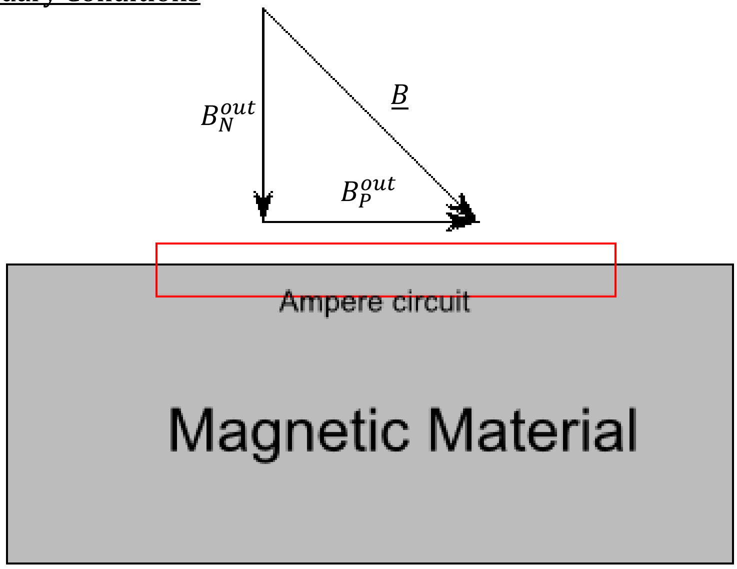
To find the behaviour of B_p consider the Amperian circuit as shown
Ampere's law $\oint \underline{B} d\underline{l} = \mu_0 \; (no\ current\ enclosed)$
In this case current loops normal to the board give a surface current that cuts through the amperian circuit
Surface currents contribute to ampere's law
b

Recap

28 February 2012
10:03

Applying a magnetic field to a magnetic material ⇒ surface currents

<u>Boundary Conditions</u>



As there are no magnetic monopoles, $\underline{B}$ lines can't end⇒ $\boxed{B_N^{out} = B_N^{in}}$
Consider ampere circuit as drawn.

Ampere's law tells us that $B_p^{out} \neq B_P^{in}$
If we have a surface current perpendicular to board
We can write $B_p^{in} = (1+\chi)B_p^{out}$
Where χ is the magnetic susceptibility

For a simple (linear, isotropic) material, χ is a constant.
In this case, it makes sense to define $\mu_r = 1+\chi (= constant)$
As far as maxwell's equations are concerned in simple magnetic materials, $\mu_0 \to \mu_0\mu_r$ then ignore material (c.f. simple dielectric: $\epsilon_0 \to \epsilon_0\epsilon_r$ then ignore material)
In general the response of the material is non-linear and $\chi \to \chi(B)$

Types of magnetic material

28 February 2012
10:18

The circulating electrons in any material constitute current loops. A single current loop is a very short solenoid ⇒ generates a magnetic dipole. Most electrons are in counter rotating pairs ⇒ dipoles cancel out.
An unpaired electron ⇒ net magnetic dipole.

1. Diamagnets
   No permanent dipoles- "dire" magnets
2. Paramagnets
   Contain permanent dipoles
   Dipoles align with applied field
3. Ferromagnets
   Contain permanent dipoles
   At low T, dipoles are not free to rotate

Diamagets
These materials contain no permanent dipoles
Do contain pairs of counter rotating electrons which act as current loops (little circuits)
If we apply a magnetic field we will change the flux linked to the circuit/orbiting electron
By faraday & lenz's, this induces an EMF which opposes the change
Dodgy classical argument: One electron speeds up, the other slows down. This produces a small net magnetic dipole which acts against the applied field and reduces it
Quantum Mechanically: the electron wave functions are distorted with the same effect
The magnetic field inside is reduced ⇒ $\chi < 0$
Diamagnetic effect is small. Typically $\chi \sim -10^{-5}$
The diamagnetic effect is present for all materials- in para&ferromagnets it is completely swamped by alignment effects

Paramagnets
These contain permanent dipoles which are free to align under the competing effects of
1) The applied field [tries to align dipoles]
2) Thermal agitation [tries to disorder system]

For low $|\underline{B}|$ the response is linear
At high$|\underline{B}|$ all dipoles are aligned → magnetic response "saturates"
On the linear (low $|\underline{B}|$) regime, large T ⇒ more agitation ⇒ less alignment ⇒ smaller χ
In fact $\chi \sim c/T, c \sim constant$

Ferromagnets
High T: above "curie point" of material
   Material behaves as a paramagnet
Low T
   Dipoles are "sticky"
   - dipoles resist alignment.
   - If aligned by a big field tend to stay aligned when filed removed
     ⇒these are permanent magnets

Again, once all the dipoles are aligned, the material "saturates" and $\chi \to 0$
   Peak for iron ⇒ $\chi \sim 10^4$

Another dodgy field
Textbooks like to define $\underline{B} = \mu_0 \mu_r \underline{H}$
Recall boundary condition $B_p^{in} = \mu_r B_p^{out}$ in terms of $\underline{H}$: $H_p^{in} = H_p^{out}$

Electrostatics

01 March 2012
12:09

Consider static $\underline{E}$ with $\underline{B} = \underline{j} = 0$
Maxwell in this setting

$$\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0}, \underline{\nabla} \times \underline{E} = 0$$

Other two satisfied

Consider

$$V_p^{\underline{AB}} = -\int_{\underline{A}\ (along\ path\ P)}^{\underline{B}} \underline{E} d\underline{l}$$

<Path from A to B with paths p and p'>

Now consider the related quantity

$$V_{p'}^{\underline{A.B}} = -\int_{\underline{A}(path\ p')}^{\underline{B}} \underline{E} d\underline{l}$$

Does the path matter?

$$V_p^{\underline{AB}} - V_{p'}^{\underline{A.B}} = -\left[\int_{\underline{A}\ (P)}^{\underline{B}} \underline{E} d\underline{l} + \int_{\underline{A}(p')}^{\underline{B}} \underline{E} d\underline{l}\right]$$

$$= \oint_{P-P'} \underline{E} d\underline{l} = -\int_S (\underline{\nabla} \times \underline{E}) d\underline{S} = 0$$

(as $(\underline{\nabla} \times \underline{E}) = 0$)

$\oint_{p-p'} \square$ =integral along closed path $\underline{A} \rightarrow \underline{B}$ along P, then $\underline{B} \rightarrow \underline{A}$

S is surface bounded by closed path

So we have

$$V_p^{\underline{AB}} - V_{p'}^{\underline{A.B}} = 0$$

i.e. $V^{\underline{AB}}$ is path independent as long as $(\underline{\nabla} \times \underline{E}) = 0$ $\underline{A}$&$\underline{B}$ enter as the end points of the integration range

Now we can define the potential V(x) via

$$V^{\underline{AB}} = V(\underline{B}) - V(\underline{A}) = -\int_{\underline{A}}^{\underline{B}} \underline{E} d\underline{l}$$

(*)

This defines the electrostatic potential.

We have the usual ambiguity about where V=0, sending $V(\underline{x}) \rightarrow V(\underline{x}) + const$ still satisfies (x)

<u>Consider a special case</u>

Let $\underline{B} = \underline{A} + \delta * \hat{\underline{x}}$

$\delta =$ small parameter

$\hat{\underline{x}}$= unit vector in $\underline{x}$ direction

Using (*)

$$V(\underline{A} + \delta\hat{\underline{x}}) - V(\underline{A}) = -\int_{\underline{A}}^{\underline{A}+\delta\hat{\underline{x}}} \underline{E} d\underline{l}$$

As δ is small, $\underline{E}$ is constant in the region of interest

$$\left(\int \underline{E} d\underline{l} \rightarrow \delta \underline{E} \hat{\underline{x}}\right)$$

$$V(\underline{A} + \delta\hat{\underline{x}}) - V(\underline{A}) = -\delta \underline{E} * \hat{\underline{x}} = -\delta E_x$$

$E_x = x\ cmpt\ of\ \underline{E}$

Rewriting

$$E_x = -\left(\frac{V(\underline{A} + \delta\hat{\underline{x}}) - V(\underline{A})}{\delta}\right) \rightarrow -\frac{\delta V}{\delta x}$$

As $\delta \rightarrow 0$

We have

$E_x = -\frac{\delta V}{\delta x}$

Similarly for y & z directions

$\underline{E} = (E_x, E_y, E_z) = -\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}\right) = -\underline{\nabla} V$

$\Rightarrow \boxed{\underline{E} = -\underline{\nabla} V}$

As $\underline{\nabla} \times (\underline{\nabla}\mathrm{V}) = 0$

For any V

$\Rightarrow \underline{\nabla} \times \underline{E} = 0$ is guaranteed if $\underline{E} = -\underline{\nabla} V$

The remaining Maxwell eqn is

$\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0}$

Substitute in $\underline{E} = -\underline{\nabla} V$

$\Rightarrow \nabla^2 V = -\frac{\delta}{\epsilon_0}$

<u>Uniqueness</u>

If we specify V on a closed surface S and we specify ρ in the volume bounded by S then

$\nabla^2 V = -\delta/\epsilon_0$

Has a unique solution in the volume bounded by S

<u>Proof</u>

Consider two solutions $V_a \& V_b$

$\Rightarrow \nabla^2 V_a = -\frac{\rho}{\epsilon_0}, \nabla^2 V_b = -\frac{\rho}{\epsilon_0}$

In the volume

And $V_a|_S = V_b|_S$ as they both satisfy the boundary conditions

Let $D = V_b - V_a$ and consider

$\Gamma = \int_{volume} dV \underline{\nabla} (D\underline{\nabla}D)$

Firstly use divergence theorem

$\Gamma = \int_S d\underline{S}(D\underline{\nabla}D) = 0$ as $D = 0$ on surface

Secondly, take Γ and apply product rule

$\Gamma = \int_{volume} dV\left((\underline{\nabla}D) * (\underline{\nabla}D) + D\nabla^2 D\right)$

Now

$\nabla^2 D = \nabla^2(V_b - V_a) = -\frac{\delta}{\epsilon_0} - \left(-\frac{\delta}{\epsilon_0}\right) = 0$

This side gives

$\Gamma = \int_{volume} dV\ (\underline{\nabla}D) * (\underline{\nabla}D)$

Divergence theorem approach told us

$\Gamma = 0$, putting both sides together

$\Gamma = 0 = \int_{volume} dV\ (\underline{\nabla}D)\ (\underline{\nabla}D) = \int_{volume} |\underline{\nabla}D|^2$

$|\underline{\nabla}D|^2 \geq 0.$

$If \int_{volume} dV |\underline{\nabla}D|^2 = 0$

Then $\underline{\nabla}D = 0$ everywhere in the volume

We have $\underline{\nabla}D = 0$ everywhere in the volume and $D = 0$ on the surface

Two together $\Rightarrow D = 0$ everywhere in volume

$\Rightarrow V_a = V_b$ everywhere in volume

$\Rightarrow$ there can be only one solution

Recap: Electrostatics

13 March 2012
10:04

Static $\underline{E}$, no $\underline{B}$ crucially then, $\underline{\nabla} \times \underline{E} = 0$ then we can use a potential i.e. set $\underline{E} = -\underline{\nabla} V$

$$\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0}$$

Then becomes

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$

(*)

If V is given on some closed surface S, there is a unique solution to (*) in the volume bounded by S

1. Known charge distribution

   For a single point charge q,

   $$V = \frac{q}{4\pi\epsilon_0 r}$$

   r=distance from the charge

   Let the charge be at position (a,b,c) and look at the potential at (x,y,z)

   Using Pythagoras,

   $$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

   $$\Rightarrow V(x,y,z) = \frac{q}{4\pi\epsilon_0\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

   Exercise

   Show that this solves $\nabla^2 V = -\rho/\epsilon_0$

   2 steps:

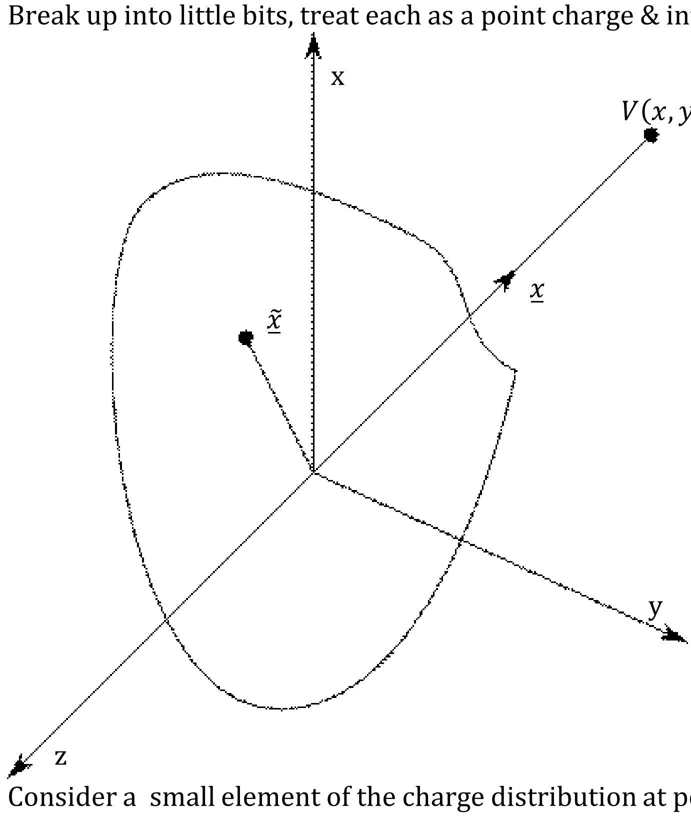
   a. Show that $\nabla^2 V = 0$ for $(x,y,z) \neq (a,b,c)$ [good practice]
   b. Use divergence theorem in a small region around (a,b,c) to show that the normalisation correct [more subtle]

   Collection of point charges

   Just sum the potentials from each one

   Continuous charge distribution

   Break up into little bits, treat each as a point charge & integrate



   Consider a small element of the charge distribution at position $\underline{\tilde{x}}$

   The volume $dV = d\tilde{x}d\tilde{y}d\tilde{x}$

   The charge in this volume is $\rho(\underline{\tilde{x}})dV$

   Viewing this element as a point charge it gives a contribution to the potential of

$$\frac{\rho(\underset{\sim}{\tilde{x}})dV}{4pi\epsilon_0|\underline{x}-\underset{\sim}{\tilde{x}}|} \equiv \frac{\rho(\tilde{x},\tilde{y},\tilde{z})d\tilde{x}d\tilde{y}d\tilde{z}}{4\pi\epsilon_0\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z})^2}}$$

To find the total potential, we add up all the contributions i.e. integrate over charge distrubution

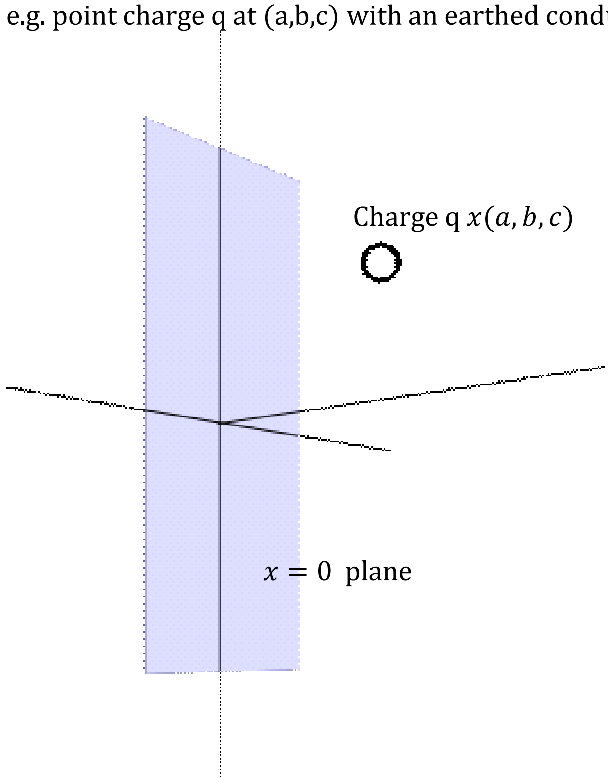
$$V(\underline{x}) = \int_{\text{charge distribution}} \frac{\rho(\underset{\sim}{\tilde{x}})dV}{4pi\epsilon_0|\underline{x}-\underset{\sim}{\tilde{x}}|}$$

$\underline{x}$ is fixed

Integrate over $\underset{\sim}{\tilde{x}}$ ← positions of the charges

2. Charge distribution PLUS boundary conditions

e.g. point charge q at (a,b,c) with an earthed conducting plate in x=0 plane



Method of images

If we can "mock up" the boundary conditions without changing the charge distribution in the region of interest, by uniqueness the solution to the new problem in the region of interest is THE solution to both problems

e.g. in our example, we're interested in the x>0 region ⇒ don't change charge distribution in x>0 region

Remove earthed plate and "mock up" boundary conditions

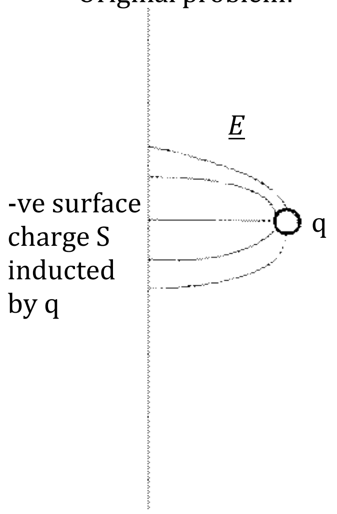
Consider placing a charge -q at (-a,b,c)

The potential due to two charges is

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+(z-c)^2}}\right]$$

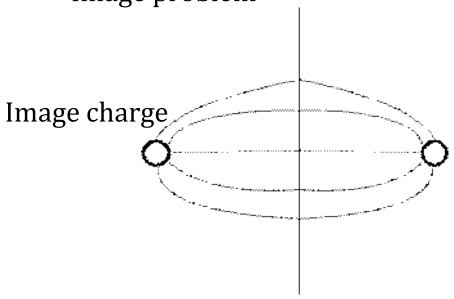
This gives $V(0,y,z)=0$

$V \to 0 \; as \; x, y, z \to \infty$
In the x>0 region, we just have charge q at (a,b,c)
By uniqueness, this is the potential in the x>0 region of the original problem
Original problem:



Electric field lines must hit plate at 90 deg as $\underline{E} = 0$ in the plate and $E_{parallel}$ is continuous

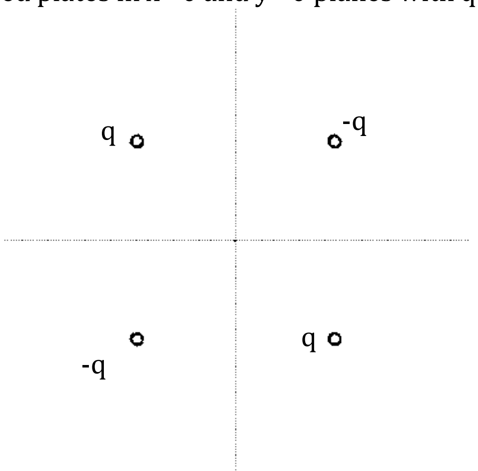
Image problem



The image charges must be outside the region of interest

Other examples

1. Earthed plates in x=0 and y=0 planes with q at (a,b,c) in lower right quadrant



Magnetic Potential

14 March 2012
11:39

Helmoltz's theorem

1. Any vector field ($\underline{H}$) is uniquely determined by giving its divergence & curl in some region and its normal component over the boundaries

    i.e. we need to specify $\underline{\nabla} * \underline{H}$ and $\underline{\nabla} \times \underline{\mathrm{H}}$ in the volume and normal component on surface

    Let $\underline{\nabla}\underline{\mathrm{H}} = \mathrm{S}$

    $\underline{\nabla} \times \underline{\mathrm{H}} = \underline{\mathrm{c}}$

2. If S and $\underline{c}$ vanish at ∞, $\underline{H}$ can be written as

    $\underline{H} = -\underline{\nabla}\phi + \underline{\nabla} \times \mathrm{A}$

    ϕ =scalar function

    $\underline{A}$ =vector function

    We have already seen this in electrostatics: with $\underline{\nabla} \times \underline{\mathrm{E}} = 0$ we used $\underline{E} = -\underline{\nabla}\mathrm{V}$

    For the magnetic field, $\underline{\nabla}.\underline{\mathrm{B}} = 0$ always as there are no magnetic monopoles.

    Helmholtz $\Rightarrow$ we can always write $\underline{B} = \underline{\nabla} \times \underline{\mathrm{A}}$

    $\underline{A}$ is a vector potential

Why use $\underline{A}$?

- $\underline{\nabla}\underline{\mathrm{B}} = \underline{\nabla} * (\underline{\nabla} \times \underline{\mathrm{A}}) = 0$ is automatic
- It will help when we talk about radiation
- Charged particles couple to $\underline{A}$ in quantum mechanics

As $\underline{\nabla} * \underline{\mathrm{B}} = 0$ we can use $\underline{B} = \underline{\nabla} \times \underline{A}$

$\underline{A}$= magnetic vector potential (gauge)

Useful mathematically

Charged particles couple to $\underline{A}$ in quantum mechanics/quantum field theory

→Aharonov-Bohm effect

Electromagnetic potentials

As $\underline{\nabla} * \underline{\mathrm{B}} = 0$, Helmholtz (2) lets us write $\underline{B} = \underline{\nabla} \times \underline{\mathrm{A}}$

Now $\underline{\nabla} \times \underline{\mathrm{E}} = -\dot{\underline{\mathrm{B}}}$

(maxwell 3)

So we have $\underline{\nabla} \times \underline{\mathrm{E}} = -\frac{\mathrm{d}}{\mathrm{dt}}(\underline{\nabla} \times \underline{\mathrm{A}}) = -\underline{\nabla} \times \left(\dot{\underline{\mathrm{A}}}\right)$

$\Rightarrow \underline{\nabla} \times \left(\underline{\mathrm{E}} + \dot{\underline{\mathrm{A}}}\right) = 0$

Using Helmoltz (2) we can write $\underline{E} + \dot{\underline{A}} = -\underline{\nabla}\mathrm{V}$

Generalisation of electrostatic potential to any situation

In any situation we have

$$\boxed{\underline{B} = \underline{\nabla} \times \underline{\mathrm{A}}}$$

$$\boxed{\underline{E} = -\dot{\underline{A}} - \underline{\nabla}\mathrm{V}}$$

In terms of these potentials, $\underline{\nabla} * \underline{\mathrm{B}} = 0$ and $\underline{\nabla} \times \underline{\mathrm{E}} = -\dot{\underline{\mathrm{B}}}$ come for free (2/4 maxwell equations solved for free)

Gauge ambiguity persists

Consider

$$\begin{pmatrix} V \\ \underline{A} \end{pmatrix} \to \begin{pmatrix} V \\ \underline{A} \end{pmatrix} + \begin{pmatrix} -\frac{\delta}{\delta t} \\ \underline{\nabla} \end{pmatrix} f$$

f=arbitrary scalar function

$\underline{B}$ starts as $\underline{\nabla} \times \underline{A}$

$\underline{B} \to \underline{\nabla} \times (\underline{A} + \underline{\nabla}f) = \underline{\nabla} \times \underline{A}$ as $\underline{\nabla} \times \underline{\nabla}f = 0$ for any f$\Rightarrow$ $\underline{B}$ is unchanged

Similarly $\underline{E}$ starts as $-\dot{\underline{A}} - \underline{\nabla}V$

$$\underline{E} \to -\frac{\delta}{\delta t}(\underline{A} + \underline{\nabla}f) - \underline{\nabla}\left(v - \frac{\delta}{\delta t}\right) = -\frac{\delta}{\delta t}\underline{A} - \underline{\nabla}V - \frac{\delta}{\delta t}\underline{\nabla}f + \underline{\nabla}\frac{\delta f}{\delta t} = -\frac{\delta}{\delta t}\underline{A} - \underline{\nabla}V$$

$\Rightarrow$ $\underline{E}$ is unchanged

"gauge transformation"

$$\begin{pmatrix} V \\ \underline{A} \end{pmatrix} \rightarrow \begin{pmatrix} V \\ \underline{A} \end{pmatrix} + \begin{pmatrix} -\frac{\delta}{\delta t} \\ \underline{\nabla} \end{pmatrix} f$$

Leaves $\underline{E}$ and $\underline{B}$ unchanged

We have a choice of which V and $\underline{A}$ to use to give a particular $\underline{E}$&$\underline{B}$:

A good choice makes the maths easier

The physics is always the same

To do this we make a "gauge choice" by imposing a "gauge condition"

A gauge condition is some extra constraint (of our choice) which isn't invariant under the gauge transformation

Example: for radiation problems, the most convenient gauge choice is

$$\underline{\nabla} * \underline{A} + \frac{1}{c^2}\dot{V} = 0$$

Is this allowed? i.e. is it invariant under

$$\begin{pmatrix} V \\ \underline{A} \end{pmatrix} \rightarrow \begin{pmatrix} V \\ \underline{A} \end{pmatrix} + \begin{pmatrix} -\frac{\delta}{\delta t} \\ \underline{\nabla} \end{pmatrix} f$$

$$\underline{\nabla} * \underline{A} + \frac{1}{c^2}\dot{V} \rightarrow \underline{\nabla} * (\underline{\mathrm{A}} + \underline{\nabla}\mathrm{f}) + \frac{1}{\mathrm{c}^2}\left(\dot{v} - \frac{\delta^2}{\delta t^2}f\right) - \underline{\nabla} * \underline{\mathrm{A}} + \frac{\dot{\mathrm{V}}}{c^2} + \left(\nabla^2 - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}\right)f$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}\right)f \neq 0 \text{ for arbitrary f}$$

$\Rightarrow$ our gauge condition isn't invariant$\Rightarrow$ it is a suitable gauge condition

Radiation

Using V and $\underline{A}$, $\underline{\nabla} * \underline{B} = 0$ and $\underline{\nabla} \times \underline{E} = -\dot{\underline{B}}$ come for free, leaving $\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0}$ and $\underline{\nabla} \times B = \mu_0(\underline{j} + \epsilon_0\dot{\underline{E}})$

We rewrite these in terms of V and $\underline{A}$ using our gauge choice $\underline{\nabla} * \underline{A} + \frac{\dot{V}}{c^2} = 0\ (GC)$

$$\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0} \rightarrow \underline{\nabla} * \left(-\dot{\underline{A}} - \underline{\nabla}V\right) = \rho/\epsilon_0$$

(Using $\underline{E} = -\dot{\underline{A}} - \underline{\nabla}V$)

$$\Rightarrow \left(\frac{\ddot{V}}{c^2} - \nabla^2 V\right) = \frac{\rho}{\epsilon_0}$$

(Using (GC), $\underline{\nabla} * \dot{\underline{A}} = -\frac{\ddot{V}}{c^2}$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\delta^2}{\delta t^2} - \nabla^2\right)V = \frac{\rho}{\epsilon_0}$$

(in this gauge)

Wave equation for V sourced by the charge density

We also have $\underline{\nabla} \times \underline{B} = \mu_0(\underline{j} + \epsilon_0\dot{\underline{E}})$

Using definitions, this gives

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \mu_0\left(\underline{j} + \epsilon_0\frac{\delta}{\delta t}(-\dot{\underline{A}} - \underline{\nabla}V)\right)$$

$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) \rightarrow \underline{\nabla}(\underline{\nabla} * \underline{A}) - \nabla^2\underline{A}$ (using vector result)

$$\Rightarrow \mu_0\epsilon_0\ddot{\underline{A}} - \nabla^2\underline{A} = \mu_0\underline{j} - \underline{\nabla}(\underline{\nabla} * \underline{A}) - \mu_0\epsilon_0\underline{\nabla}\dot{V}$$

$$-\underline{\nabla}(\underline{\nabla} * \underline{A}) - \mu_0\epsilon_0\underline{\nabla}\dot{V} = -\underline{\nabla}(\underline{\nabla} * \underline{A} + \mu_0\epsilon_0\dot{V})$$

$$-\underline{\nabla}\left(\underline{\nabla} * \underline{A} + \frac{1}{c^2}\dot{V}\right) = 0 \text{ By (GC)}$$

In this gauge,

$$\left(\frac{1}{c^2}\frac{\delta^2}{\delta t^2} - \nabla^2\right)\begin{pmatrix} V \\ \underline{A} \end{pmatrix} = \begin{pmatrix} \frac{\rho}{\epsilon_0} \\ \mu_0\underline{j} \end{pmatrix}$$

Wave equations for $\dot{V}$ and $\underline{A}$ sourced by charge and current densities respectively

Recap

27 March 2012
10:09

We can express E and B in terms of potentials V and A

$\underline{B} = \underline{\nabla} \times \underline{A}, \underline{E} = -\underline{\dot{A}} - \underline{\nabla} V$

Then

$\underline{\nabla} * \underline{B} = 0 \;\; and\; \underline{\nabla} \times \underline{E} = -\underline{\dot{B}}\; come\; for\; free$

<u>Gauge Ambiguity</u>

E and B are unchanged if

$$\begin{pmatrix} V \\ \underline{A} \end{pmatrix} \to \begin{pmatrix} V \\ \underline{A} \end{pmatrix} + \begin{pmatrix} -\frac{\delta}{\delta t} \\ \underline{\nabla} \end{pmatrix}$$

\`\`\`\`\`

Example

For radiation problems, gauge condition

$$\underline{\nabla} * \underline{A} = \frac{1}{c^2}\dot{V} = 0$$

Is useful

In <u>THIS gauge</u>, remaining maxwell eqns give

$$\underline{\nabla} * \underline{E} = \frac{\rho}{\epsilon_0} \to \left(\frac{1}{c^2}\frac{\delta^2}{\delta t^2} - \nabla^2\right)V = \frac{\rho}{\epsilon_0}$$

$$\underline{\nabla} \times \underline{B} = \mu_0(\underline{j} + \epsilon_0 \underline{E}) \to \left(\frac{1}{c^2}\frac{\delta^2}{\delta t^2} - \nabla^2\right)\underline{A} = \mu_0 \underline{j}$$

Take the eqn for V

$$\left(\frac{1}{c^2}\frac{\delta^2}{\delta t^2} - \nabla^2\right)V = \frac{\rho}{\epsilon_0}$$

And consider the static case, i.e. $\frac{\delta^2}{\delta t^2}V = 0$, we're left with $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

We are back to electrostatics and we know the solution so we do an integral over the charge distribution

Some small element at $(\tilde{x}, \tilde{y}, \tilde{z})$, charge is $\rho(\tilde{x}, \tilde{y}, \tilde{z})d\tilde{V}$

$d\tilde{V} = volume\ element\ d\tilde{x}d\tilde{y}d\tilde{z}$

The contribution of this element to the potentiaal is

$$\frac{\rho(\underline{\tilde{x}})d\tilde{V}}{4\pi\epsilon_0 d} \equiv \frac{\rho(\underline{\tilde{x}})d\tilde{V}}{4\pi\epsilon_0|\underline{x} - \underline{\tilde{x}}|}$$

Summing over all the elements

$$V(\underline{x}) = \int \frac{\rho(\underline{\tilde{x}})d\tilde{x}d\tilde{y}d\tilde{z}}{4\pi\epsilon_0|\underline{x} - \underline{\tilde{x}}|}$$

We really need

$$\left(\frac{1}{c^2}\frac{\delta^2}{\delta t^2} - \nabla^2\right)V = \frac{\rho}{\epsilon_0}$$

If the charges move, the information about any changes takes time to propagate to us ⇒ we "see" the charge distribution as it was a light travel time ago. If fact the full solution is obtained by integrating over the sources with the appropriate time lag

$$\Rightarrow V(\underline{x}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\underline{\tilde{x}}, t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}\right)d\tilde{x}d\tilde{y}d\tilde{z}}{|\underline{x} - \underline{\tilde{x}}|}$$

Similarly

$$\Rightarrow \underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}\left(\underline{\tilde{x}}, t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}\right)d\tilde{x}d\tilde{y}d\tilde{z}}{|\underline{x} - \underline{\tilde{x}}|}$$

Comments

- These are called "retarded" potentials
- Our arguments about look back in time motivates use of $t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}$ but doesn't

proof the expressions given

- You can verify that the given expressions solve the equations [exercise for the enthusiastic]

Dipole Radiation

27 March 2012
10:32

In the gauge where

$$\underline{\nabla} * \underline{A} = \frac{1}{c^2}\dot{V} = 0$$

$\underline{A}$ & V satisfy

$$\left(\frac{1}{c^2}\frac{\delta^2}{\delta t^2} - \nabla^2\right)\begin{pmatrix} V \\ \underline{A} \end{pmatrix} = \begin{pmatrix} \frac{\rho}{\epsilon_0} \\ \mu_0 \underline{j} \end{pmatrix}$$

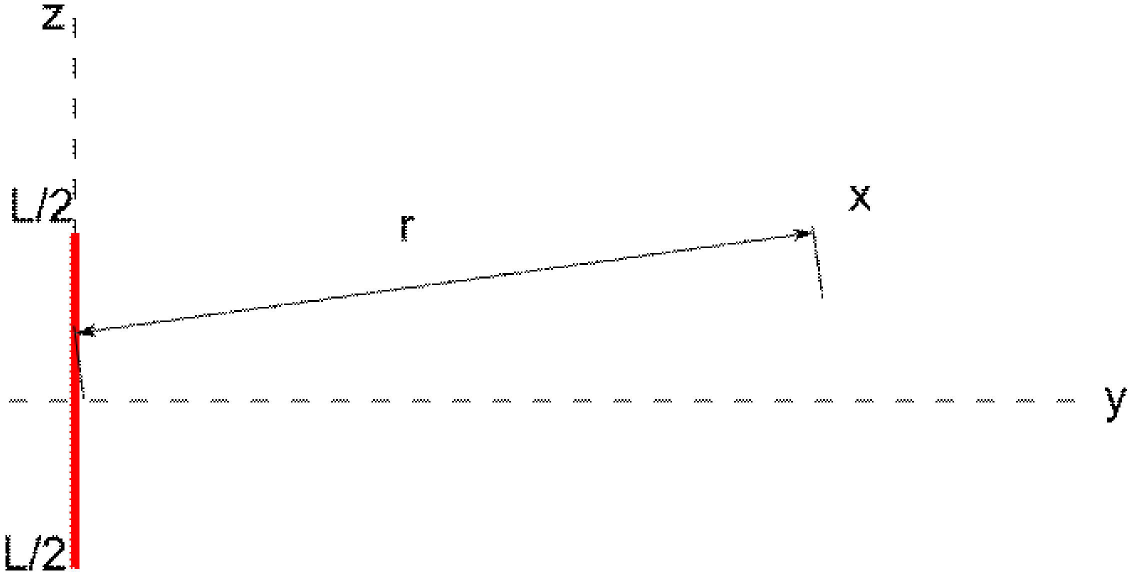
These are solved by using "retarded" integrals

$$V(\underline{x}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\underline{\tilde{x}}, t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}\right) d\tilde{x}d\tilde{y}d\tilde{z}}{|\underline{x} - \underline{\tilde{x}}|}$$

$$\underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}\left(\underline{\tilde{x}}, t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}\right) d\tilde{x}d\tilde{y}d\tilde{z}}{|\underline{x} - \underline{\tilde{x}}|}$$

Consider an aerial of length L carrying an oscillating current $I = I_0 \cos \omega t$
The aerial is centred at the origin, parallel to z-axis
We will consider distances $\gg L$ [we call this a short areal $L \ll$ distances of interest]

$r \gg L$



We know

$$\underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}\left(\underline{\tilde{x}}, t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}\right) d\tilde{x}d\tilde{y}d\tilde{z}}{|\underline{x} - \underline{\tilde{x}}|}$$

As $r \gg L$ we can take $|\underline{x} - \underline{\tilde{x}}| = r$ for all elements of the areal
If the wire has a cross-sectional area a, the current density

$$\underline{j} = \frac{I_0}{a}\cos \omega t\, \underline{\hat{z}}$$

So

$$\underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi}\int_{\substack{\text{volume} \\ \text{of wire}}} \frac{\underline{j}\left(\underline{\tilde{x}}, t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}\right) \underline{\hat{z}} d\tilde{x}d\tilde{y}d\tilde{z}}{|\underline{x} - \underline{\tilde{x}}|}$$

Setting $|\underline{x} - \underline{\tilde{x}}| = r$

$$\underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi}\int_{\substack{\text{volume} \\ \text{of wire}}} \frac{\underline{j}\left(\underline{\tilde{x}}, t - \frac{r}{c}\right) \underline{\hat{z}} d\tilde{x}d\tilde{y}d\tilde{z}}{r}$$

$$\Rightarrow \underline{A}(\underline{x},t) = \frac{\hat{\underline{z}}\mu_0}{4\pi}\frac{I_0}{a}\frac{\cos\omega\left(t-\frac{r}{c}\right)}{r}\int_{\substack{\text{Volume}\\ \text{of wire}\\ (=aL)}} d\tilde{x}d\tilde{y}d\tilde{z} = \frac{\mu_0}{4\pi}I_0 l\cos\omega\left(t-\frac{r}{c}\right)\hat{\underline{z}}$$

$$\underline{A} = \frac{\mu_0}{4\pi r}I_0 l\cos\omega\left(t-\frac{r}{c}\right)\hat{\underline{z}}$$

Magnetic field

Using

$$\underline{B} = \underline{\nabla}\times\underline{A} = \begin{bmatrix}\delta_y A_z - \delta_z A_y\\ \delta_z A_x - \delta_x A_z\\ \delta_x A_y - \delta_y A_x\end{bmatrix}$$

For this case, only $A_z \neq 0$

Also $\underline{A} = \underline{A}(r)$ so using the chain rule,

$$\underline{B} = \begin{pmatrix}\delta_y A_z\\ -\delta_x A_z\\ 0\end{pmatrix} = \frac{dA_z}{dr}\begin{pmatrix}\frac{\delta r}{\delta y}\\ -\frac{\delta r}{\delta x}\\ 0\end{pmatrix} = \frac{\mu_0 I_0 l}{4\pi}\left\{\frac{\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right)}{r} - \frac{\cos\omega\left(t-\frac{r}{c}\right)}{r^2}\right\}\begin{pmatrix}\frac{y}{r}\\ -\frac{x}{r}\\ 0\end{pmatrix}$$

(Using $r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{\delta r}{\delta x} = \frac{x}{\sqrt{\square}} = \frac{x}{r}$

$\dfrac{\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right)}{r}$ Dominates at smaller-"induction field"

$\dfrac{\cos\omega\left(t-\frac{r}{c}\right)}{r^2}$ Dominates at larger -"Radiation field"

Comments: $\underline{B}$ is perpendicular to $\hat{\underline{Z}}$ and $\underline{r}$

Radiation field decays as 1/r

Electric field

We will obtain the electric field from V

As we haven't specified the charge distribution, we can't do the integral to get V. instead we use the gauge condition

$$\underline{\nabla}.\underline{A} + \frac{1}{c^2}\dot{V} = 0 \Rightarrow \dot{V} = -c^2\underline{\nabla}.\underline{A}$$

$$= -c^2\frac{\delta A_z}{\delta z}\ \textit{as only}\ A_z \neq 0$$

$$= -c^2\left(\frac{dA_z}{dr}\right)\frac{dr}{dt}$$

$$= -\frac{c^2\mu_0 I_0 L}{4\pi}\left\{\frac{\omega}{L}\frac{\sin\omega\left(t-\frac{r}{c}\right)}{r} - \frac{\cos\omega\left(t-\frac{r}{c}\right)}{r^2}\right\}\frac{z}{r}$$

Integrating with respect to time (dropping any integration constant)

$$V = -\frac{c^2\mu_0 I_0 L}{4\pi}\left\{-\frac{\cos\omega\left(t-\frac{r}{c}\right)}{cr} - \frac{\sin\omega\left(t-\frac{r}{c}\right)}{\omega r^2}\right\}\frac{z}{r}$$

Now we can use

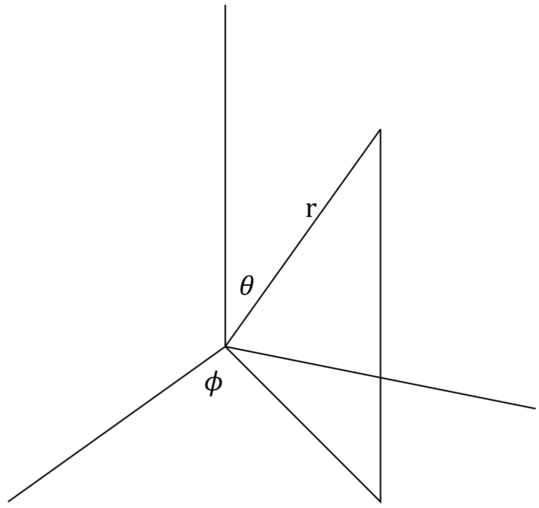
$$\underline{E} = -\dot{\underline{A}} - \underline{\nabla}V$$

$$\left(\text{remember } \underline{A} = \frac{\mu_0 I_0 l}{4\pi}\frac{\cos\omega\left(t-\frac{r}{c}\right)}{r}\hat{\underline{z}}\right)$$

$$\underline{E} = \begin{bmatrix}0\\ 0\\ \frac{\mu_0 I_0 l}{4\pi}\frac{\sin\omega\left(t-\frac{r}{c}\right)}{r}\end{bmatrix} + \frac{c^2\mu_0 I_0 l}{4\pi}\left\{-\sin\omega\left(t-\frac{r}{c}\right)\frac{\omega}{c^2 r} + O\left(\frac{1}{r^2}\right)\right\}\begin{pmatrix}\frac{x}{r}\\ \frac{y}{r}\\ \frac{z}{r}\end{pmatrix}\frac{z}{r}$$

$$\Rightarrow \underline{E} = \frac{\mu_0 I_0 l\omega}{4\pi r}\sin\omega\left(t-\frac{r}{c}\right)\begin{bmatrix}-\frac{x^2}{r^2}\\ -\frac{yz}{r^2}\\ 1-\frac{z^2}{r^2}\end{bmatrix} + O\left(\frac{1}{r^2}\right)$$

This is cleaner in polar coordinates



$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\underline{B} = \frac{\mu_0 I_0 l\omega}{4\pi cr}\sin\omega\left(t-\frac{r}{c}\right)\sin\theta\begin{pmatrix}\sin\phi\\ -\cos\phi\\ 0\end{pmatrix}$$

$$\underline{E} = \frac{\mu_0 I_0 l\omega}{4\pi r}\sin\omega\left(t-\frac{r}{c}\right)\sin\theta\begin{pmatrix}-\cos\theta\cos\phi\\ -\cos\theta\sin\phi\\ \sin\theta\end{pmatrix}$$

$$\begin{pmatrix}\sin\phi\\ -\cos\phi\\ 0\end{pmatrix} = \hat{\underline{e}}_\phi$$

$$\begin{pmatrix}-\cos\theta\cos\phi\\ -\cos\theta\sin\phi\\ \sin\theta\end{pmatrix} = -\hat{\underline{e}}_\theta$$

$\underline{E}$&$\underline{B}$ are orthogonal to each other and direction of travel

Energy flux is given by Poynting vector

$$\frac{1}{\mu_0}\underline{E}\times\underline{B} \rightarrow \text{Parallel to direction of travel}$$

Averaging over a cycle, the energy flux is

$$\frac{1}{\mu_0}\left(\frac{\mu_0 I_0 \omega l}{4\pi r}\right)^2\frac{\sin^2\theta}{2c} = \bar{p}$$

To find the total power radiated, integrate energy flux over a sphere of radius r

$$\text{Total Power} = \int r^2\sin\theta\, d\theta d\phi dr\, \bar{p} = \frac{I_0^2 l^2\omega^2\mu_0}{12\pi c}$$

Short Aerial

(length L)

$I = I_0\cos\omega t$

Integrate over $\underline{j} \rightarrow \underline{A}$

Use $\underline{B} = \underline{\nabla}\times\underline{A}$ to find

$$\underline{B} = \left(\frac{\mu_0 I_0 l\omega}{4\pi cr}\right) * \sin\theta\sin\omega\left(t-\frac{r}{c}\right)(-\hat{\underline{e}}\phi)$$

Use Gauge condition to find V

Then $\underline{E} = -\dot{\underline{A}} - \underline{\nabla}V$

$$\Rightarrow \underline{E} = \frac{\mu_0 I_0 l\omega}{4\pi r}\sin\theta\sin\omega\left(t-\frac{r}{c}\right)(-\hat{\underline{e}}\theta)$$

$\underline{E}$&$\underline{B}$ are mutually perpendicular and both perpendicular to direction of travel.

⇒We have an EM wave moving radially out from aerial

Average energy flux per cycle

$$\bar{p} = \frac{1}{\mu_0}\left(\frac{\mu_0 I_0 l\omega}{4\pi r}\right)^2\frac{\sin^2\theta}{2c}$$

Factor of $\sin^2\theta$ ⇒ most power comes out normal to aerial, nothing parallel to aerial

Total power radiated

$$= \int r^2 \sin\theta \, d\theta d\phi \bar{p} = \frac{I_0^2 l^2 \omega^2 \mu_0}{12\pi c}$$

This is often written in terms of dipoles. Consider the current flowing between two charge "reservoirs"

When the reservoirs hold charge $\pm Q$ this has a dipole moment $D = QL = D_0 \sin\omega t \; (defines\ D_0)$

Differentiate wrt time

$$\dot{D} = \dot{Q}L = \omega D_0 \cos\omega t = IL = (I_0 \cos\omega t)L \Rightarrow \boxed{I_0 L = \omega D_0}$$

Using $I_0 L = \omega D_0$ and $c^2 = 1/\epsilon_0\mu_0$, the power becomes

$$\boxed{\frac{D_0^2\omega^4}{12\pi c^3 \epsilon_0}}$$

This is proportional to (Dipole moment)^2 and ω^4

Not expected to derive this in an exam, BUT should remember this result.

Recap

28 March 2012
11:07

In the gauge where

$$\underline{\nabla} * \underline{A} = \frac{1}{c^2}\dot{V} = 0$$

$\underline{A}$ & V satisfy

$$\left(\frac{1}{c^2}\frac{\delta^2}{\delta t^2} - \nabla^2\right)\begin{pmatrix} V \\ \underline{A} \end{pmatrix} = \begin{pmatrix} \dfrac{\rho}{\epsilon_0} \\ \mu_0 \underline{j} \end{pmatrix}$$

These are solved by using "retarded" integrals

$$V(\underline{x}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\underline{\tilde{x}}, t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}\right) d\tilde{x}d\tilde{y}d\tilde{z}}{|\underline{x} - \underline{\tilde{x}}|}$$

$$\underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}\left(\underline{\tilde{x}}, t - \frac{|\underline{x} - \underline{\tilde{x}}|}{c}\right) d\tilde{x}d\tilde{y}d\tilde{z}}{|\underline{x} - \underline{\tilde{x}}|}$$

Classical Field Theory

24 April 2012
10:21

Is there any principle underlying Maxwell's equations?
Yes- if we use action principles and Lagrangian (as in dynamics II last year)

<u>Dynamics in the Lagrangian Language</u>

In 1-dimension, the path x(t) taken by a particle travelling between $x_1(t_1)$ and $x_2(t_2)$ extremises the integral

$$\int_{t_1}^{t_2} L(x, \dot{x}, t) dt$$

Where L is the Lagrangian: $L = Ke - Pe$

[NB: total energy=Ke+Pe]

The mathematical solution to extremising the integral is given by the Euler-Lagrange equations: solution when

$$\frac{\delta L}{\delta x} = \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right)$$

Where $\frac{\delta}{\delta x}$ means a partial derivative wrt x

In this example x happens to be a coordinate

Mathematically, however, it is just the function of t that extremises the integral (path)

For E&M we need to treat space & time on equal footing.

We generalise

$$\int_{t_1}^{t_2} dt \rightarrow \int dt\, d^3x$$

i.e. integral over spacetime

Lagrangian

$$L(x, \dot{x}, t) \rightarrow \mathcal{L}\left(f, \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z}, \frac{\delta f}{\delta t}, x, y, z, t\right)$$

x= "path"
$\dot{x}$ = derivatives of path wrt coords
t=coordinate
$\mathcal{L}$ =lagrangian density
f=path
$\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z}, \frac{\delta f}{\delta t}$= derivatives of "path" w.r.t. coordinates
x, y, z, t=coords.

We need to extremise

$$\int dt d^3x \mathcal{L}(\dots)$$

$\mathcal{L}$ is given by the physics

The mathematical solution is given by Euler-Lagrange

$$\frac{\delta \mathcal{L}}{\delta f} = \frac{\delta}{\delta x_i}\left(\frac{\delta \mathcal{L}}{\delta\left(\frac{\delta f}{\delta x^i}\right)}\right)$$

The pair of indices ⇒ summation convention. In this case, Einstein Summation convention where we sum over one "upstairs" and one "downstairs" index e.g.

$$G_a H^a \equiv \sum_{a=0}^{3} G_a H^a$$

Here G_a and H^a are 4-vectors

Indices are raised and lowered using the metric

$$H_a = g_{ab}H^b, G^a = g^{ab}G_b$$

Where

$$g^{ab} = diag(-1, +1, +1, +1) = g_{ab}$$

But what is $\mathcal{L}$

We know(in sensible $\epsilon = \mu_0 = 1$ units) the energy density in the $\underline{E}$ and $\underline{B}$ fields is $\frac{1}{2}(E^2 + B^2)$

Compare $E_{tot} = Ke + Pe$ before and $L = Ke - Pe$

The lagrangian density is just the difference: $\mathcal{L} = \frac{1}{2}(E^2 - B^2)$

This doesn't have derivatives in ⇒ work with $\underline{A}$ and V instead

We're trying to encode Maxwell's eqns in an action principle
Use euler-lagrange to find the functions that extremize ******
$\underline{E}$ and $\underline{B}$ can be "packaged into the field strength tensor

$$\underline{\underline{f}} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} (*)$$

This is a tensor as it has nice transformation properties under Lorentz-transformations

Working with 4-vectors

$A^\mu = (V, A_x, A_y, A_z), \mu = 0,1,2,3$

$$\delta_\mu = \left(\frac{\delta}{\delta t}, \frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}\right)$$

Then

$$f_{\mu_\nu} = \delta_\mu A_\nu - \delta_\nu A_\mu$$

[exercise: show that this agrees with (*)]

The lagrangian is

$$\mathcal{L} = \frac{1}{4} f_{\mu\nu} f^{\mu\nu}$$

Where

$$f^{\mu\nu} = g^{\mu\alpha} f_{\alpha\beta} g^{\beta\nu}$$

Apply Euler-Lagrange

$\Rightarrow \delta_\mu(\delta^\mu A^\nu - \delta^\nu A^\mu) = 0$

$\nu = - \Rightarrow \underline{\nabla}.\underline{E} = 0$

$\nu = 1,2,3 \rightarrow \underline{\nabla} \times \underline{B} = \underline{\dot{E}}$

Vacuum Maxwell's eqns

Exercise: work through this

Remember $\underline{\nabla}.\underline{B} = 0$ and $\underline{\nabla} \times \underline{E} = -\underline{\dot{B}}$ come for free as we're using $V, \underline{A}$ (remember c=1 here)

Comments

1. First step towards standard model
2. Beware signs- different metric conventions scatter minus signs around
3. This last section (i.e. "classical field theory") is non-examinable as started yesterday

Revision Lecture

02 May 2012
11:06

Gauge condition/invariance

$$\begin{pmatrix} V \\ \underline{A} \end{pmatrix} \rightarrow \begin{pmatrix} V \\ \underline{A} \end{pmatrix} + \begin{pmatrix} -\frac{\delta}{\delta t} \\ \underline{\nabla} \end{pmatrix}$$

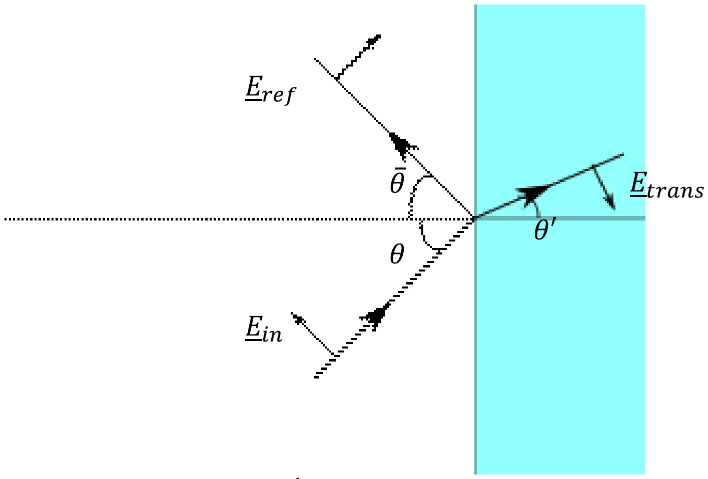
$$\underline{\nabla} * \underline{A} + \frac{1}{c^2}\dot{V} = 0$$

$$\underline{\nabla} * \underline{A} + \frac{1}{c^2}\dot{V} \rightarrow \underline{\nabla}.(\underline{A} + \underline{\nabla}\mathrm{f}) + \frac{1}{c^2}\frac{\delta}{\delta t}\left(\mathrm{V} - \frac{\delta}{\delta t}\mathrm{f}\right)$$

$$= \left(\underline{\nabla A} + \frac{1}{c^2}\dot{v}\right) + \left(\nabla^2 - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}\right)f$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}\right)f \neq 0\ for\ arbitrary\ f$$

Snell's law etc



$$\underline{E}_{in} = \underline{E}_0^{in} \exp\left[-i\omega t + \frac{i\omega}{c}(\cos\theta x + \sin\theta y)\right]$$

$$\underline{E}_{ref} = \underline{E}_0^{ref} \exp\left[-i\bar{\omega} t + \frac{i\bar{\omega}}{c}(\cos\bar{\theta} x + \sin\bar{\theta} y)\right]$$

$$\underline{E}_{trans} = \underline{E}_0^{trans} \exp\left[-i\omega' t + \frac{i\omega'}{c'}(\cos\theta' x + \sin\theta' y)\right]$$

Parallel component of $\underline{E}$ is continuous
Just outside $(x = 0_-)$ = just inside $(x = 0_+)$

$$|\underline{E}_0^{in}|\cos\theta \exp\left[-i\omega t + \frac{i\omega}{c}(\cos\theta x + \sin\theta y)\right] + |\underline{E}_0^{ref}|\cos\bar{\theta} \exp\left[-i\bar{\omega} t + \frac{i\bar{\omega}}{c}(\cos\bar{\theta} x + \sin\bar{\theta} y)\right]$$

$$= |\underline{E}_0^{trans}|\cos\theta' \exp\left[-i\omega' t + \frac{i\omega'}{c'}(\cos\theta' x + \sin\theta' y)\right]$$

This holds on the whole surface at any time ⇒ for all t & for all
Consider y=0 form is (const)$e^{-i\omega t}\overline{(\mathrm{const})}e^{-\bar{\omega} t} = (\mathrm{const}')e^{-i\omega' t}$
For this to hold for all, we must have $\omega = \bar{\omega} = \omega'$
Similarly y dependence

$$(\mathrm{const})\exp\left[\frac{i\omega}{c}\sin\theta y\right] + \overline{(\mathrm{const})}\exp\left[\frac{i\bar{\omega}}{c}\sin\bar{\theta} y\right] = (\mathrm{const}')\exp\left[\frac{i\omega'}{c'}\sin\theta' y\right]'$$

To hold for all y, we need

$$\frac{\sin\theta}{c} = \frac{\sin\bar{\theta}}{c} = \frac{\sin\theta'}{c'}$$

Reflection law

$$\theta = \bar{\theta}$$

Snells law

$$\frac{\sin\theta}{\sin\theta'} = \frac{c}{c'}$$

Energy density

$$\frac{\epsilon_0}{2}\underline{E}.\underline{E} + \underline{B}.\frac{\underline{B}}{2\mu_0}$$