MECHANICS AND SPECIAL RELATIVITY

1. SPACE AND TIME IN NEWTONIAN DYNAMICS.

In Newtonian dynamics, particles move along trajectories in absolute space. Time is also absolute, and external - all observers measure identical time.

The arena for dynamics is three-space 3-dim Euclidean space. Time is an external parameter.

3-dim Euclidean space is the setting where conventional geometric rules apply, eg. shortest distance between points is a straight line, Pythagoras theorem, sum of angles in triangle = 180°, etc.

A point P is described by coordinates x, y, z relative to the origin O. $\vec{OP}$ is the vector $\underline{x} = (x, y, z)$.

Trajectories:-

A trajectory is a path through space. This is described by a function giving the particle position as a function of time, ie. $\underline{x}(t)$. A trajectory is also called a 'world-line'.

The dynamical equations of motion are differential equations ($\underline{F} = m\dfrac{d^2\underline{x}(t)}{dt^2}$) which determine the trajectories $\underline{x}(t)$.

With no forces acting, a particle follows a straight line path through space, moving with uniform velocity, $\dfrac{d\underline{x}}{dt} = \underline{v} = \text{const}$.

Metric:-

Two points (x, y, z) and $(x+dx, y+dy, z+dz)$ are separated by a distance ds, where

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= (dx\ dy\ dz) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

This is just Pythagoras' rule. This is only true for a flat space. A curved space (eg in 2-dim, the surface of a sphere) satisfies a different rule.

The distance rule is called the metric relation. The matrix $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ above is called the metric. It specifies that the 3-dim space is Euclidean.

Vectors, etc.

We can define basic geometric quantities on this space, eg. scalars, vectors, etc. They are defined by how they transform under rotations (see below).

The position vector $\underline{x} = (x, y, z)$ is the simplest example of a vector. We can specify a general vector $\underline{V}$ by its components along the x, y, z axes :- $\underline{V} = (V_x, V_y, V_z)$

An equivalent notation is to use indices $i = 1, 2, 3$ for the components :- $\underline{V} = (V_x, V_y, V_z) = (V_1, V_2, V_3)$, and write the position vector as $\underline{x} = (x, y, z) = (x_1, x_2, x_3)$.

This 'index' notation is very powerful. For the scalar product, we have

$$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i .$$

$$= \sum_i \sum_j \delta_{ij} u_i v_j$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ is the Kronecker delta symbol.

The (Einstein) summation convention is to omit the Σ signs and sum over repeated indices. So we just write

$$\underline{u} \cdot \underline{v} = \delta_{ij} u_i v_j$$

We can write this in matrix form as $(u_1\ u_2\ u_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

This is the same as the distance rule. δ_{ij} is the metric. The metric interval is $ds^2 = \delta_{ij}\, dx_i\, dx_j$.

To write the vector product, we need the Levi-Civita antisymmetric symbol ϵ_{ijk}, where

$\epsilon_{123} = 1 \qquad \epsilon_{213} = -1$ & cyclic perms

$\epsilon_{ijk} = 0$ if any 2 indices are the same.

The i^{th} component of the vector product is:-

$$(\underline{u}\times\underline{v})_i = \epsilon_{ijk}\, u_j v_k$$

So, eg. $$(\underline{u}\times\underline{v})_1 = \epsilon_{123} u_2 v_3 + \epsilon_{132} u_3 v_2 + 0$$

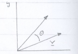
$$= u_2 v_3 - u_3 v_2$$

The index notation is very powerful and generalises easily to curved spaces and different dimensions.

Rotations:-

Under a rotation in 3 dim, $v_i \to R_{ij} v_j$ where R_{ij} are components of the 3×3 rotation matrix (Arfken, p200).

This is simple to see in 2 dim.



$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{matrix } R \text{, cpts } R_{ij}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The rotation matrix is orthogonal $R^T R = 1$

In index notation, $R^T_{ij} R_{jk} = R_{ji} R_{jk} = \delta_{ik}$.

This rotation property defines what we mean by a vector. A scalar is invariant under rotations, $f \to f$

ame of reference :-

A frame of reference is a system of measuring position (and time) set up by a set of observers with identical motions in space.

In general, the description of events (laws of physics) will appear different in different frames of reference.

Eg. one frame of reference (S) could be fixed on the earth, while another frame (S') could be fixed on an airplane.

Each frame has its own set of coordinates :- (x, y, z) for S and (x', y', z') for S'. It is automatically assumed that $t' = t$, ie. time is universal and independent of motion.

Newtonian dynamics

The first postulate of Newtonian dynamics gives a special role to the class of frames corresponding to observers in uniform (constant velocity) motion. These are called Inertial Frames.

…late (1) :- The laws of dynamics (~~physics~~) are …me in all inertial frames.

These …(100) …tial frames are equivalent.

So, e.g., we cannot distinguish the physics that holds in a smoothly-flying airplane from physics on earth.

This postulate is hugely important. It is a relativity principle - it admits that there is no absolute rest frame. Only relative motion is meaningful.

Postulate (2) :- Since inertial frames are equivalent, the effect of interactions, forces, can only be to produce an acceleration. Thus $\underline{F} = m \frac{d^2 \underline{x}}{dt^2}$.

Forces do not produce changes in velocity - they change acceleration.

Postulate (3) :- Dynamics takes place in the arena of flat (Euclidean) 3-dim space. It follows that the laws of dynamics are invariant under Rotations and Translations.

A deep theorem (Noether's theorem) shows that :-

Invariance under translations $\Rightarrow$ conservation of momentum.
" " rotations $\Rightarrow$ " - angular momentum.

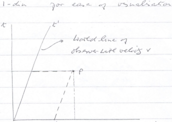
These three postulates are a more precise formulation of the usual elementary form of Newton's laws.

(The whole structure applies to dynamics. Electromagnetism, i.e. Maxwell's equations, are not invariant under Galilean transformations.)

Galilean transformations:-

We need to relate measurements made in different inertial frames.

Look first at 1-dim for ease of visualisation.



Read off coords. by projection parallel to axes.

By assumption, $t' = t$.

The position of P in frame S is x, in frame S' is x'.
Velocity of S' relative to S is v.

$$\left.\begin{aligned} &\text{Evidently,} & x' &= x - vt \\ &\text{together with} & t' &= t \end{aligned}\right\}$$

$$\text{In 3-dim,} \quad \left.\begin{aligned} \underline{x}' &= \underline{x} - \underline{v}\,t \\ t' &= t \end{aligned}\right\}$$

These are called Galilean transformations.

To check consistency with postulate (1), we should find that the force law is invariant under Galilean transformations:-

$$\underline{F} = m\frac{d^2\underline{x}}{dt^2} = m\frac{d^2\underline{x}'}{dt'^2}$$

since $t' = t$ and $\frac{d\underline{x}'}{dt} = \frac{d\underline{x}}{dt} - \underline{v}$

$$\Rightarrow \frac{d^2\underline{x}'}{dt^2} = \frac{d^2\underline{x}}{dt^2}$$

For completeness, we note also here the Translation and Rotation transformations:-

Translation:-

$$\underline{x}' = \underline{x} + \underline{a}$$
$$t' = t$$

Rotation

$$x_i' = R_{ij} x_j$$
$$t' = t$$

where R_{ij} is the 3-dim rotation matrix:-

$$R_{ij} = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ +\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

(Arfken, p200, makes a different choice corresponding to Euler angles.)

SPACE AND TIME IN SPECIAL RELATIVITY.

The Newtonian picture of space and time is radically changed in Special Relativity (Einstein 1905).

Special relativity is based on the following postulates:-

Postulate (1A):- The laws of physics are the same in all inertial frames.

Postulate (1B):- The speed of light (c) is the same in all inertial frames.

These postulates immediately imply a radical revision of our 'common sense' concepts of space and time.
As in Newtonian dynamics, we will later add two further 'postulates' when we introduce relativistic dynamics.

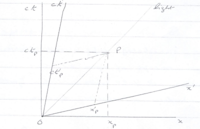
Notice that Postulate (1B) introduces a new fundamental constant of nature, the speed of light c.

(This is a new scale, setting a fundamental velocity.
Newtonian dynamics has no such scale – all velocities are equivalent.
Newtonian dynamics will be recovered as an approximation to special relativity for velocities $v \ll c$.

This is analogous to Quantum Mechanics, which also introduces a new fundamental constant, h. Newtonian dynamics is the limit of QM for scales (energy × time) $\gg h$.)

Postulate (1A) is really the same as in Newtonian dynamics. (It is usually emphasised that in special relativity it is applied to all the laws of physics, including electromagnetism. This is only the statement that Maxwell's electromagnetism is not compatible with the Galilean transformations of Newtonian dynamics, but does respect the Lorentz transformations of special relativity (see later.))

Now concentrate on Postulate (1B). To make this true, we have to modify the space-time diagram as follows:-



S' has velocity v relative to S.

The only way to make the speed of light the same in frame S (coords x, t) and frame S' (coords x', t') is to allow $t' \neq t$.

This is a huge result! It means that different observers in relative motion have different <u>times</u>!

Equations applied to S:

First Try:- (Wrong!)

$$\left.\begin{aligned} x' &= x - vt = x - \tfrac{v}{c}ct \quad \text{(as before)} \\ ct' &= ct - vx/c \end{aligned}\right\} \quad (1)$$

This makes the transformation symmetrical for space and (time × c)

Then the velocity is $\dfrac{x'}{t'} = \dfrac{x - vt}{t - vx/c^2} = \left(\dfrac{x}{t} - v\right)\left(1 - \dfrac{x}{t}\dfrac{v}{c^2}\right)^{-1}$

If $c \to \infty$, this is the 'common-sense' velocity addition rule.

Measuring a light signal OP, $\dfrac{x}{t} = c \Rightarrow \dfrac{x'}{t'} = (c - v)\left(1 - \dfrac{v}{c}\right)^{-1}$

$= c$.

So by making $t' \neq t$, and making the transformation symmetric, we satisfy Postulate (1B).

But what about Postulate (1A)? S' has velocity v relative to S, but this means S has velocity $-v$ relative to S'. The transformation must satisfy this. So we must have

$$\left.\begin{aligned} x &= x' + vt' \\ ct &= ct' + \frac{vx'}{c} \end{aligned}\right\} \quad (2)$$

However, if we invert eqs (1), we get instead,

$$x = \left(1 - v^2/c^2\right)^{-1}\left(x' + vt'\right)$$
$$ct = \left(1 - v^2/c^2\right)^{-1}\left(ct' + \frac{vx'}{c}\right)$$

So this First Try is Wrong! Incompatible with Postulate (1A).

Lorentz Transformations

The correct transformations are as follows. We can preserve agreement with Postulate (1B) even if we change the scale of x' and t', as long as it involves the same factor.

So write
$$\left.\begin{aligned} x' &= \gamma\,(x - vt) \\ ct' &= \gamma\left(ct - \frac{vx}{c}\right) \end{aligned}\right\} \quad (1)$$

where γ is a function of v^2.

Postulate (1B) is satisfied. Postulate (1A) requires

$$\left.\begin{aligned} x &= \gamma\,(x' + vt') \\ ct &= \gamma\left(ct' + \frac{vx'}{c}\right) \end{aligned}\right\} \quad (2)$$

Inverting (1) we now have,

$$x = \gamma^{-1}\left(1 - v^2/c^2\right)^{-1}\ (x' + vt')$$
$$ct = \gamma^{-1}\left(1 - v^2/c^2\right)^{-1}\ \left(ct' + \frac{vx'}{c}\right)$$

This must agree with (2). This requires,

$$\gamma = \left(1 - v^2/c^2\right)^{-1/2}.$$

Now, <u>both</u> Postulates (1A) and (1B) are satisfied.

We have now derived the correct space and time transformations. They are the generalisation of the Galilean transformations which are compatible with the postulates of special relativity.

They are called the Lorentz transformations:-

1 dim:- $x' = \gamma(x - vt)$

$t' = \gamma(t - vx/c^2)$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

3 dim:- $x' = \gamma(x - vt)$

$t' = \gamma\left(t - \dfrac{v x}{c^2}\right)$

$y' = y$

$z' = z$

$$\Rightarrow \underline{x}' = \gamma(\underline{x} - \underline{v}t) + (\gamma - 1)\frac{\underline{v}\times(\underline{v}\times\underline{x})}{v^2}$$

(They were originally found before Einstein, as the transformations under which the Maxwell electromagnetism equations are invariant. So electromagnetism needs no modification - it is already special relativistic!)

Since the Lorentz transformations mix space and time, we have to change our concepts. The fundamental arena is now 4-dimensional spacetime, rather than 3-dim space and separate universal time.

The next step is to describe the geometry of spacetime.

Minkowski Spacetime :-

Both x and t are changed as we transform from frame S to frame S'. Does anything remain invariant?

Calculate $-c^2t'^2 + x'^2 = -c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2 + \gamma^2(x-vt)^2$

$$= -c^2t^2 + x^2$$

using $\gamma^2 = (1-v^2/c^2)^{-1}$.

So although frames S and S' assign different values of space and time coordinates to an event, they both agree on the combination $(-c^2t^2 + x^2)$. We will call this the spacetime interval. In 3 dim, it is $(-c^2t^2 + \underline{x}\cdot\underline{x})$

This is the spacetime generalisation of the Pythagoras rule in 3-dim Euclidean space, where the distance $(x^2+y^2+z^2)$ is invariant when we compare different frames.

So formalise this, we define Minkowski spacetime as the 4-dim spacetime with metric rule,

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

$$\Rightarrow \quad ds^2 = \begin{pmatrix} cdt & dx & dy & dz \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{metric}} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}.$$

Even more formally, define $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

The metric relation is,

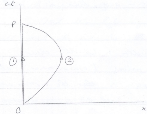
$$ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu \qquad \text{(sum over } \mu, \nu\text{)}$$

where $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ is the metric.

Also $x^\mu = L^\mu{}_\nu x^\nu$ $\qquad ds^2 = g_{\mu\nu} L^\mu{}_\lambda L^\nu{}_\rho \, dx^\lambda dx^\rho$

$\Rightarrow g_{\mu\nu} L^\mu{}_\lambda L^\nu{}_\rho = g_{\lambda\rho} \quad \Leftrightarrow \quad L^T g L = g$. Compare $R^T 1 R = 1$

Now compare 2 paths in Minkowski spacetime:-



The spacetime interval along a path is $\int_{\text{path}} ds$

Clearly, $\int_{\text{path 1}} ds > \int_{\text{path 2}} ds$

(Do this explicitly for straight line paths - Tutorial sheet).

For the observer following a given path, the entire spacetime interval is interpreted as being time. (In his frame, he has not moved.)

So if we compare the 2 paths ① and ②, the observer following path ① from 0 to P will measure a longer time than the observer following path ②.

This is the famous Astronaut (or Clock or Twin) 'Paradox' in Special Relativity.

There is no paradox. The resolution is simply that the paths are different - and elapsed time is a path-dependent quantity.

The spacetime picture makes this clear. It is less obvious if we think naively (common-sense) about the motion. One common misconception is that the motions are symmetric - they are not, because path ② is accelerated. It cannot be followed by a single inertial frame (co-moving frame). So we cannot, for example, re-draw the above diagram showing path ② straight wrt x', t' axes where x', t' are measured by the astronaut. This is not an inertial frame.

MEASUREMENTS OF SPACE AND TIME

We now study a number of special cases illustrating space and time measurements in special relativity.

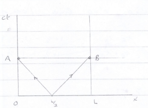
3.1 SIMULTANEITY:-

It follows from the SR spacetime diagram that the lines of simultaneity ($t=0$ or $t'=0$) are different in different frames S and S'.

Two events, spatially separated, which take place at the same time in frame S occur at different times in frame S' (ie. according to a moving observer.)

Simultaneity is not an absolute property - it depends on the frame of reference.

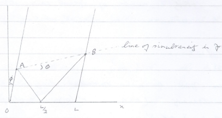
We can look at this operationally. Suppose we have a rod of length L. Send two light signals from the centre point. They will reach the ends at the same time:-



$t_A = t_B$

Events A and B are simultaneous in frame S.
This gives an operational way of defining simultaneity.

Now suppose the rod is moving, with velocity v. In the rest frame of the rod, simultaneity is defined the same way.



Light signal received at ends of rod at A and B.

By definition, $t'_A = t'_B$.

In S frame:- $x_A = vt_A = \frac{L}{2} - ct_A \Rightarrow ct_A = \frac{L}{2}\cdot\frac{1}{1+v/c}$

$x_B = vt_B + L = \frac{L}{2} + ct_B \Rightarrow ct_B = \frac{L}{2}\cdot\frac{1}{1-v/c}$

So $t_A \neq t_B$, ie. events not simultaneous in frame S.

Compare angles of $x'=0$, $t'=0$ lines (see diagram):-

$$\tan\theta = \frac{c(t_B - t_A)}{x_B - x_A} = \frac{t_B - t_A}{t_B + t_A} = \frac{1+v/c - (1-v/c)}{1+v/c + 1 - v/c} = v/c$$

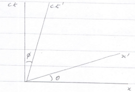
$$\tan\phi = \frac{x_A}{ct_A} = v/c$$

Angles are equal. This confirms the "skew-axes" form for frame S' (coords t', x').

Check for Lorentz transformations :-

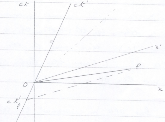
$t' = \gamma(t - vx/c^2)$ $\qquad$ $t' = 0 \iff \tan\theta = \frac{ct}{x} = v/c$

$x' = \gamma(x - vt)$ $\qquad$ $x' = 0 \iff \tan\phi = \frac{x}{ct} = v/c$



3.2 FASTER THAN LIGHT / BACKWARDS IN TIME :-

Suppose it is possible to travel faster than light. (The totally hypothetical particle with this property is called a tachyon.) What would be the consequences?



For frame S' shown, $t'_p < 0$

For frame S, $t_p > 0$.

Time

So a signal moving faster than light is moving forward in time in some frame (like S) but backward in time in other frames (like S').

Suppose signal is "warp factor" λ, ie. speed λc. From diagram this is backwards in time in frames where the x' axis is at a bigger angle than OP, ie.

$$\tan\theta = \frac{v}{c} > \frac{1}{\lambda}$$

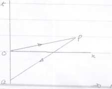
Check from Lorentz transformation:-

$$t' = \gamma(t - vx/c^2)$$
$$x' = \gamma(x - vt)$$

$\Rightarrow$ $t' < 0$ if $t - \frac{vx}{c^2} < 0$, ie. $\frac{x}{ct} > \frac{c}{v}$

or $\lambda > \frac{c}{v}$.

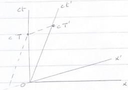
The problem with this is that if the motion OP is possible, then (by Postulate 1A) there must be the possibility of a signal PQ. (Faster than light, forward in time in some frame, just like OP in frame S).



But OPQ is a signal to the past world line of O.
This is a violation of causality and is presumed to be impossible
$\Rightarrow$ No tachions in special relativity

TIME DILATION :-

Now compare the measurements of time differences in different frames.



Frame S :- time interval = T. (x = const.)

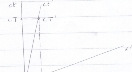
Frame S' :- time interval $= T' = \gamma\left(T - \frac{v\overset{0}{x}}{c^2}\right)$

$= \gamma T$

Since $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$, $T' > T$

So the moving frame assigns a bigger time difference to the interval $t = 0 \rightarrow t = T$ at $x = 0$.

Vice-versa :-



Frame S' :- time interval $= T'$ $\quad (x' = \text{const})$

Frame S :- time interval $= T = \gamma\left(T' + \frac{v\,\overset{\to 0}{x'}}{c^2}\right)$

$= \gamma T'$

So $\quad T > T'$

This is the same result :-

the time difference assigned in a moving frame is bigger than the time interval between 2 events at the same place in a stationary frame.
This is time dilation.

Another way of saying this is that the minimum time interval between 2 events is measured in the frame in which the 2 events are at the same position.

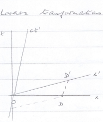
Later, we will look at experiments which verify this directly, eg. μ decay, Hafele-Keating clocks.

LENGTH CONTRACTION:-

When we discuss space interval measurements, we must be careful to define very carefully what is being measured.

Space dilation:-

There is an effect essentially identical to time dilation. This is obvious from the space/time symmetry of the Lorentz transformation:-

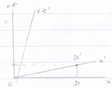


Frame S :- distance interval $= D$ (at $t = \text{const}$)

Frame S' :- distance interval $= D' = \gamma\left(D - v\overset{\to 0}{t}/c^2\right)$

$= \gamma D > D$

Vice-versa:-



Frame S' :- distance interval $= D'$ (at $t' = \text{const}$)

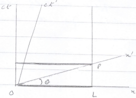
Frame S :- distance interval $= D = \gamma\left(D' + v\overset{\to 0}{t'}/c^2\right)$

$= \gamma D' > D'$

Length contraction:-

The measurement of distance intervals described above does not however correspond to the operational definition of a length measurement.

Consider a rod of length L at rest in frame S.



$\tan\theta = \frac{v}{c}$ (see sect. 3.1)

When the length of the rod is measured in frame S', the positions of the ends of the rod are measured at the same time wrt frame S'.

So to measure the length of the rod in frame S', the left-hand end is measured at event O, while the right-hand " " " " P.

The "length" is defined as the spatial separation (in S') between points O and P, ie

$$L' = x'_P - x'_O$$

In frame S, $\quad x_P = L \qquad ct_P = L\tan\theta = L\,v/c$

frame S', using Lorentz transformation,

$$x'_p = \gamma(x_p - vt_p) \qquad t'_p = \gamma(t_p - vx_p/c^2)$$

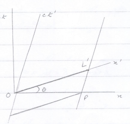
$$\Rightarrow \quad L' = \gamma(L - Lv^2/c^2) \qquad , \text{ since } t'_p = 0 \Rightarrow t_p = \frac{vx_p}{c^2}$$

$$= \gamma^{-1} L$$

That is $\quad L' < L$

So the moving frame S' measures a <u>shorter</u> length for the rod compared to the rest frame S.
This is <u>length contraction</u>.

Vice-versa:- Rod moving, ie. in frame S', with length L'.



$\tan\theta = v/c$

Frame S measures length L of rod as the space interval OP between the ends measured <u>at the same time wrt S</u>,

ie. $\quad L = x_p - x_o$

Lorentz transformation:-

$$x_p = \gamma\left(x'_p + v t'_p\right) \qquad t_p = \gamma\left(t'_p + v x'_p/c^2\right)$$

Since $t_p = 0$ $\Rightarrow$ $t'_p = -v x'_p/c^2$

Clearly $x'_p = L'$

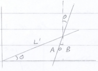
So $\Rightarrow$ $x_p = \gamma\left(L' - \frac{v^2}{c^2} L'\right)$

$$= \gamma^{-1} L'$$

So we find $L = \gamma^{-1} L' < L'$

Again, the length measurement is shorter in the frame moving relative to the rod.

(Geometrically:- See diag.



$-ct'_p = A = \frac{B}{\cos\theta}$ where $B = L'\sin\theta$

$\Rightarrow$ $-ct'_p = L'\tan\theta = L' v/c$, as above.)

Pole and garage 'paradox':-

Length contraction gives rise to another of the popular 'paradoxes' of special relativity.

A man with a pole of length L' runs into a garage which is shorter than the pole.

In the garage frame, pole length is $L = \gamma^{-1} L' < L'$ so the pole fits inside the garage!

But in the man's frame, the garage is Lorentz contracted. Is there a paradox? (Problem sheet 3, (7))

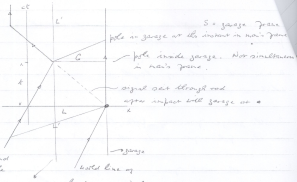
The situation has to be analysed carefully. The front end of the pole hits the garage end wall and stops, sending a signal of finite speed ($< c$) through the pole.

Viewed in the garage frame, the pole then re-expands to a length bigger than the garage.

In the man's frame, the events are the same. It is not relevant to use the length contraction formula here, since the instant (simultaneous in garage frame) when the pole is inside is not an equal-time measurement in the man's frame. At the instant (simultaneous in man's frame) the pole is in the garage (see diag.) it is compressed.

An important aspect of this is that totally rigid bodies don't exist in special relativity, otherwise instantaneous transmission would be possible.

The spacetime diagram is:-



Length of rod = L'. Lorentz contracted length is L

Garage length if pole just gets in before "stop" signal reaches back end is G, where:-

$G = ck$ $\qquad$ $L - G = vk$

$\Rightarrow \quad L = (1 + v/c)\, G$

But Lorentz contraction formula $\Rightarrow \quad L = \gamma' L'$

So $\quad L' = \gamma(1 + v/c)\, G$

L' is the longest pole which can be fitted into garage of length G.

5 VELOCITY ADDITION RULE :-

Suppose a particle moves with velocity u in frame S. What is its velocity in frame S', moving with velocity v relative to S?

This is easily deduced from the Lorentz transformation. In 1-dim,

$$x' = \gamma(x - vt) \qquad\qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$t' = \gamma(t - vx/c^2)$$

So for constant speeds,

$$u' = \frac{x'}{t'} = \frac{x - vt}{t - vx/c^2} = \frac{u - v}{1 - uv/c^2}$$

The same formula holds in general:-

$$u' = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma(dt - v\,dx/c^2)}$$

$$\Rightarrow \quad u' = \frac{u - v}{(1 - uv/c^2)}$$

In 3 dim,

$$u'_x = \frac{u_x - v}{(1 - u_x v/c^2)} \qquad u'_y = \frac{u_y}{\gamma(1 - \frac{u_x v}{c^2})} \qquad u'_z = \frac{u_z}{\gamma(1 - \frac{u_x v}{c^2})}$$

(i) This shows that whatever we choose for v, u' is still less than c.

The formula lets us work out the relative speeds in the following situation. Particle A moves relative to the lab with speed u_1. Particle B moves relative to A with speed u_2.

$\Rightarrow$ Particle B moves in lab with speed $\dfrac{u_1 + u_2}{1 + u_1 u_2/c^2}$

which that with u_1, u_2 less than c, so is the sum.

(iii) From the rule $u' = \dfrac{u - v}{(1 - uv/c^2)}$, we can work out the corresponding combination rule for the γ factors:

$$\gamma(u') = \gamma(u)\,\gamma(v)\left(1 - uv/c^2\right)$$

(ii) This is consistent with Postulate (1b):

Set $u = c$ $\Rightarrow$ $u' = \dfrac{c - v}{1 - cv/c^2} = c$

So the speed of light (in vacuo) is the same in all inertial frames.

Stellar Aberration:-

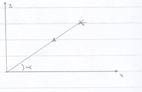
The apparent angle made by stars (for example) depends on the motion of the observer. The position of stars is found to trace a small ellipse (major axis ~ 41 arcsec) over the course of an earth orbit. This phenomenon is known as stellar aberration.

This effect can be calculated using the 3-dim velocity-addition formula.

Consider 2 frames S and S', with S' having velocity v along the x-axis. The addition formula becomes,

$$u_x' = \frac{u_x - v}{1 - u_x v/c^2} \qquad u_z' = \frac{u_z}{\gamma(1 - u_x v/c^2)}$$

In frame S, the light from the star makes an angle α to x-axis.



So "photon velocity" $\quad u_x = -c\cos\alpha$

$\qquad u_y = -c\sin\alpha$

But in the moving frame, $\quad u_x' = -c\cos\alpha'$

$\qquad u_y' = -c\sin\alpha'$

The velocity addition formula then gives,

$$\cos\alpha' = \frac{\cos\alpha + v/c}{1 + v/c \cos\alpha}$$

and

$$\sin\alpha' = \frac{\sin\alpha}{\gamma(1 + v/c\cos\alpha)}$$

Then, using the trig formula $\tan\frac{\alpha}{2} = \sin\alpha\,(1+\cos\alpha)^{-1}$ we find

$$\tan\frac{\alpha'}{2} = \left(\frac{c-v}{c+v}\right)^{1/2} \tan\frac{\alpha}{2}$$

This is the most useful form of the stellar aberration formula.

For $v \ll c$, we can expand,

$$\tan\frac{\alpha'}{2} = \tan\frac{\alpha}{2}\left(1 - \frac{v}{c} + O(v^2/c^2)\right)$$

so the correction is $O(v/c)$.

DOPPLER EFFECT

Consider light of frequency ν_0 emitted by a source. Suppose the source recedes from the observer, with velocity v. The frequency changes - the light is 'red-shifted'.

This effect occurs classically for all kinds of wave motion. However, special relativity predicts a new formula:-

Consider the time between 2 emitted pulses. This is $dt_0 = \frac{1}{\nu_0}$ in the rest frame of the source.

Because of time dilation, to the observer at rest this is $dt = \gamma \, dt_0$

However, during this time, the source has receded, so the light takes an extra time to reach the observer. This is $=$ (extra distance)$/c$ $= v/c \, dt$

i.e. the time separation of pulses measured by the observer is

$$dt_{obs} = dt + v/c \, dt$$

Since the observed frequency is $\nu_{obs} = \frac{1}{dt_{obs}}$, we find

$$\frac{\nu_0}{\nu_{obs}} = \frac{dt_{obs}}{dt_0} = \gamma \left(1 + v/c \right)$$

The relativistic correction is the factor γ.
(This is an $O(v^2/c^2)$ correction.)

The Doppler formula can be rewritten as

$$\frac{\nu_0}{\nu_{obs}} = \left(\frac{1 + v/c}{1 - v/c} \right)^{1/2}$$

<u>Transverse Doppler effect</u> :-

Because of time dilation, there is a Doppler effect even if the motion of the source has no radial component

This is due purely to the time dilation part of the above calculation, so we get

$$\left. \frac{\nu_0}{\nu_{obs}} \right|_{\substack{\text{transverse} \\ \text{Doppler}}} = \gamma$$

This gives a direct way of measuring time dilation experimentally.

EXPERIMENTS.

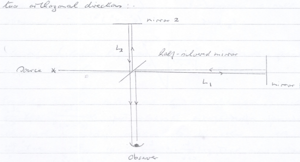
In this section we briefly discuss some experiments which test in a very direct way the picture of spacetime presented here.

Of course, the indirect tests are commonplace. All of particle physics, including the operation of the accelerators at CERN, is based on special relativity.

4.1. MICHELSON - MORLEY EXPT.

This is a classic interferometer experiment, performed originally in 1887.

The Michelson interferometer compares the motion of light in two orthogonal directions:



Light beam split. Compare rays that have followed paths L_1 and L_2

These interfere, giving fringes. A change in the length L_1 or L_2 gives a change in the fringe pattern, allowing tiny distance changes to be measured.

(Interferometers are the way gravitational wave experiments are done.)

Suppose Postulate (1B) was wrong, and light satisfied a 'common-sense' addition law for velocities.

So suppose vel of light $= c$ in some frame.

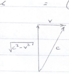
Now assume the interferometer moves at speed v relative to this :-

Then velocity of light on first half of L_1 path $= c - v$

" " " " " $= c + v$

and " L_2 path $= (c^2 - v^2)^{1/2}$

(Vector addition of velocities :-



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So, time taken for path 2 is $T_2 = \dfrac{2L_2}{(c^2-v^2)^{1/2}} = \gamma \dfrac{2L_2}{c}$

Time for path 1 is $T_1 = \dfrac{L_1}{c+v} + \dfrac{L_1}{c-v} = \dfrac{2L_1}{c\left(1 - v^2/c^2\right)} = \gamma^2 \dfrac{2L_1}{c}$

the lengths ... $T_1 \neq T_2$ (even if $L_1 = L_2$).

So a rotation of the interferometer should produce a shift in the fringes – the dependence on v should be observable.

No such effect was observed.

Interpretation (modern) :- From our point of view, this confirms Postulate (1B), that the speed of light is the same in all inertial frames.

Interpretation (historical) :- In the late 1800s, it was believed that light travelled as a disturbance in a medium called the ether. The Michelson-Morley experiment should have detected the velocity of the earth relative to the ether (i.e. find a v dependence). But instead a null result is obtained.

The only conclusions left are either (i) the ether does not exist or (ii) the ether is fixed relative to the earth.

Possibility (ii) is ruled out by the observation of stellar aberration. The pre-relativistic interpretation assumes that the earth is moving relative to the ether – the stellar aberration factor $\left(1 - \frac{v}{c}\right)$ is interpreted as a classical Doppler shift. (cf formulae in sects 3.5 and 3.6)

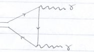
There is one last get-out for the ether – the ad-hoc suggestion by Lorentz and Fitzgerald of length contraction by a factor γ^{-1} in the

4.2 $\pi^0 \to \gamma\gamma$ AND THE SPEED OF LIGHT

The neutral pion π^0 is a meson, a bound state of a quark and antiquark. (In fact it is a 'superposition' - see QM course, $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$)

It can decay to 2 photons:

$\pi^0 \to \gamma\gamma$ $\qquad$ $\pi^0 \left\{ \begin{matrix} q \\ \bar{q} \end{matrix} \right.$



The photons should travel at the speed of light, in any frame.

If the π^0 is at rest (in the lab. frame), the speed of the γ's is indeed c.

At CERN (1964) the γ speed was measured for photons emitted by pions produced with speeds 0.99975c. The γ speed was still found to be c.

ATOMIC CLOCKS AND 'TIME DILATION'

In 1971, Hafele and Keating flew atomic clocks round the world, both east and west, on commercial airplanes and compared the times registered with clocks kept at rest.

As expected by 'time dilation' (clocks following different paths - see 'Astronaut paradox' in section 2), the clocks registered different times:-

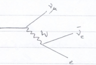
Clocks flown east: 59 nsec slow
(" " west: 273 nsec fast

Expt repeated in 15hr flights round Chesapeake Bay, Maryland in 1976
(Accuracy 1%)

4.4 MUON DECAY AND TIME DILATION

The lifetime of a muon at rest is $\sim 10^{-6}$ sec.
It decays via

$\mu \to e\ \bar{\nu}_e\ \nu_\mu$



In 1966 at CERN, muons were accelerated (in a small storage ring of 7m radius) to speeds 0.997c
Their lifetime, measured in the lab, was 12 times the lifetime at rest, as predicted by time dilation ($\gamma = 12$)

Repeated in 1978, with $\gamma = 29$, corresponding to speed 0.9994c

(For comparison, an electron in LEP (50 GeV) has $\gamma = 10^5$.)

4.5 DOPPLER EFFECT EXPTS.

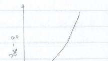
The transverse Doppler effect can be measured by detecting gamma rays from the centre of a centrifuge. (This uses the Mossbauer resonance technique. Accuracy ~ 1%)

We expect $\frac{\nu_{obs}}{\nu_0} = \gamma^{-1} = (1 - v^2/c^2)^{1/2}$

For small speeds, $v/c \ll 1$ we can approximate the γ factor by expanding up to $O(v^2/c^2)$.

$$(\Rightarrow \quad \nu_{obs} = \nu_0 \left(1 - \frac{1}{2} \frac{v^2}{c^2} + O(v^4/c^4) \right)$$

1963 expt (Kundig)
using centrifuge method.



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More recent expts using fast neon atoms as source and laser spectroscopy have verified this effect to better than 1 part in 10^5

This effect is crucial in satellite navigation systems (see Halliday & Resnick 42.11), where the satellite orbital speed is $\sim 10^4$ m/s, which gives $\gamma^{-1} - 1 \sim \frac{1}{2} v^2/c^2 = 10^{-10}$
Combined with clock accuracy of 1 part in 10^{12} allows aircraft velocities, eg. to be determined to accuracies of a few cm/sec

RELATIVISTIC DYNAMICS

In this section, we complete the formulation of special relativistic dynamics. This is most clearly formulated using the language of 4-vectors in Minkowski spacetime.

5.1 MINKOWSKI SPACETIME & 4-VECTORS:-

In section 2, we showed that the natural geometric arena for special relativity is a 4-dim spacetime, known as Minkowski spacetime.

This has coordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ and the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

with $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\quad \mu, \nu = 0,1,2,3$

i.e.

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

ds is the Lorentz-invariant spacetime interval.

In 3-dim Euclidean space (sect 1), we define 3-vectors by their properties under rotations. The prototype is the position vector:-

$$x_i \xrightarrow[R]{} x_i' = R_{ij} x_j \qquad R = \text{rotation matrix}$$

A vector is an object with components V_i which transform in the same way,

$$V_i \xrightarrow[R]{} V_i' = R_{ij} V_j$$

In 4-dim Minkowski spacetime, we define 4-vectors by their properties under rotations <u>and</u> Lorentz transformations.

(In many ways, Lorentz transformations are like rotations involving the fourth, time, coordinate. Problem sheet 5. In fact, together they form a <u>group</u> of transformations on Minkowski spacetime. Lorentz transformations are sometimes called <u>Boosts</u>)

The rotation part is the same as before.

The prototype 4-vector is the position vector x^μ in Minkowski spacetime. It has the Lorentz transformation property,

$$\begin{aligned} x &\xrightarrow[L]{} x' = \gamma\left(x - \tfrac{v}{c}ct\right) \\ y &\rightarrow y' = y \\ z &\rightarrow z' = z \\ ct &\rightarrow ct' = \gamma\left(ct - vx/c\right) \end{aligned}$$

for a L.T. in the x-direction with velocity v.

A 4-vector is a geometrical object V^μ with the same property, i.e.

$$V^1 \xrightarrow[L]{} V^{1'} = \gamma\left(V^1 - \frac{v}{c}V^0\right)$$

$$V^2 \to V^{2'} = V^2$$

$$V^3 \to V^{3'} = V^3$$

$$V^0 \to V^{0'} = \gamma\left(V^0 - \frac{v}{c}V^1\right)$$

When we formulate relativistic dynamics, we will need to find the 4-vectors corresponding to the usual vectors of dynamics, e.g. velocity, momentum, force, etc.

In Euclidean space, we can construct a scalar product from 2 vectors:-

$$\underline{u}\cdot\underline{v} = \underbrace{\delta_{ij}}_{\text{metric in 3-space}} u_i v_j = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$\underline{u}\cdot\underline{v}$ is a scalar, i.e. invariant under rotation.

Similarly, in Minkowski spacetime, we can construct a Lorentz invariant from 2 4-vectors using the metric:-

Given 4-vectors U^μ, V^μ, we define the Lorentz invariant product as

$$g_{\mu\nu} \, U^\mu V^\nu \quad = \quad -U^0 V^0 \; + \; U^1 V^1 + U^2 V^2 + U^3 V^3$$

That is, if we write $U = (U^0, \underline{u})$, we have

$$g_{\mu\nu} \, U^\mu V^\nu \quad = \quad -U^0 V^0 \; + \; \underline{u} \cdot \underline{v}$$

(It is often convenient to distinguish contravariant vectors V^μ from covariant vectors V_μ. Covariant vectors are defined as,

$$V_\mu = g_{\mu\nu} V^\nu$$

ie. $V_0 = -V^0$, $V_1 = V^1$, $V_2 = V^2$, $V_3 = V^3$

The Lorentz invariant product is then simply,

$$g_{\mu\nu} \, U^\mu V^\nu \quad = \quad U_\nu V^\nu \quad = \quad U^\mu V_\mu \qquad)$$

The Lorentz invariant formed using the same 4-vector is particularly important. It plays the role of the 'magnitude' of the vector. Since it is Lorentz invariant, it is the same measured by any inertial observer.

In fully 4-dim (covariant) notation, the Lorentz transformation is written,

$$V^{\mu'} = L^\mu{}_\nu \, V^\nu \qquad , \qquad L^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for a L.T. in x direction.

4-VELOCITY

The spacetime interval ds is Lorentz invariant.
It is useful also to define the invariant 'proper time' $d\tau$ as $d\tau^2 = -\frac{1}{c^2} ds^2$

($\int d\tau$ is the time measured by an observer following the path specified path.)

Given a 4-vector, we can clearly make another by differentiation wrt s, or τ, since the new object will have the same Lorentz transformation as the original.

If we do this for the position 4-vector x^μ we derive the velocity 4-vector, ie.

$$U^\mu = \frac{dx^\mu}{d\tau}$$

In the same way, 4-acceleration is $A^\mu = \frac{d^2x^\mu}{d\tau^2}$.

So, for a particle following a trajectory $x^\mu(\tau)$, where τ is the proper time (time measured along its worldline), we have

$$U^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt}$$

Since $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$

$$\Rightarrow \quad \frac{ds}{dt} = \left(-c^2 + \underline{u}^2\right)^{1/2}$$

where $\underline{u}$ = 3-velocity

or equivalently, $\dfrac{d\tau}{dt} = \left(1 - \dfrac{u^2}{c^2}\right)^{1/2}$

We therefore have,

$$U^\mu = \gamma(u)\,(c, \underline{u})$$

The space part of the velocity 4-vector is therefore not simply the usual Newtonian velocity - instead it contains a γ factor. The time part is chosen to give the correct Lorentz transformation.

The magnitude of the 4-velocity is a Lorentz invariant:-

$$U^2 = g_{\mu\nu} U^\mu U^\nu = \gamma^2\left(-c^2 + u^2\right)$$

$$= -c^2$$

4-MOMENTUM AND MASS.

The 4-momentum is defined as a Lorentz invariant quantity, the mass, times the 4-velocity :-

$$p^\mu = m U^\mu$$

Clearly, p^μ is a 4-vector. Its components are:-

$$p^\mu = (\gamma m c, \ \gamma m \underline{u})$$

The magnitude is Lorentz invariant - it gives the mass:-

$$p^2 = g_{\mu\nu} p^\mu p^\nu = \gamma^2 m^2 (-c^2 + u^2)$$

$$= -m^2 c^2$$

We now have to interpret these components. Writing $p^\mu = (p^0, \underline{p})$, we have the special relativistic definition of momentum,

$$\underline{p} = \gamma m \underline{u}$$

Notice that this differs from the Newtonian definition by the factor of γ.

To interpret the p^0 component, look at the small velocity approximation.

$$p^0 = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} mc = mc\left(1 + \frac{1}{2}\frac{u^2}{c^2} + O\left(\frac{u^4}{c^4}\right)\right)$$

$$\Rightarrow \quad cp^0 = mc^2 + \frac{1}{2}mu^2 + O\left(\frac{u^4}{c^4}\right)$$

The second term is simply the usual Newtonian kinetic energy.

This motivates a radical interpretation – cp^0 is to be identified as energy!

We therefore write $\qquad p^\mu = \left(\frac{E}{c}, \underline{p}\right)$

and identify E as the particle energy.

The dramatic conclusion is that the particle energy is not zero when it is at rest. There is a 'rest-energy', $E_{rest} = mc^2$.

In general, $\qquad \underline{p} = \gamma m \underline{u}, \qquad E = \gamma mc^2$

The discovery of 'rest-energy' and the formula $E = \gamma mc^2$ is one of the great triumphs of Einstein's theory.

Energy and momentum are therefore united. p^μ is called the energy-momentum 4-vector, or simply 4-momentum.

It follows immediately that energy E and momentum $\underline{p}$ depend on the frame of reference. Their Lorentz transformations are determined from the fact that P^μ is a 4-vector, ie.

$$E' = \gamma\left(E - v p^1\right)$$
$$p^{1'} = \gamma\left(p^1 - vE/c^2\right)$$
$$p^{2'} = p^2$$
$$p^{3'} = p^3$$

for a L.T. in the x-direction.

So just as space and time are mixed by a Lorentz transformation, so are energy and momentum.

As with all 4-vectors, we can find a Lorentz invariant by taking the magnitude of P^μ :-

$$g_{\mu\nu} P^\mu P^\nu = -\frac{E^2}{c^2} + \underline{p}\cdot\underline{p}$$

$$= -\gamma^2 m^2 c^2 + \gamma^2 m^2 u^2 \qquad , \quad \gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

$$= -m^2 c^2$$

The invariant is just the <u>mass</u>!

This is one of the most important formulae in relativity. In any frame,

$$E^2 - c^2 p^2 = m^2 c^4$$

("Relativistic mass" :-

In many texts, the notation $m(u) = \gamma(u)$ is introduced, simply to allow the pseudo-Newtonian formula $\underline{p} = m(u)\,\underline{u}$ to be written. This "relativistic mass" then increases with velocity.

This (in my opinion) is useless and conceptually confusing. Mass is a Lorentz invariant.)

PHOTONS :-

Photons (and possibly neutrinos) are particles with zero mass. The formulae above are then specially simple :-

Since $E^2 - c^2 p^2 = 0$,

the energy is simply $E = cp$

(This is compatible with the quantum formula $E = h\nu$ and de-Broglie wavelength $\lambda = h/p$.)

The photon 4-momentum is $p^\mu = (\frac{E}{c}, \underline{p})$ where $E^2 = c^2 p^2$.

It is a singular limit of the general formula $p^\mu = (\gamma m c, \gamma m \underline{u})$ in which $m \to 0$ and so necessarily $\gamma \to \infty$, ie. $u \to c$.
Photons of course (as any $m=0$ particle) travel at the speed of light.

POSTULATES OF RELATIVISTIC DYNAMICS.

We can now complete the formulation of relativistic dynamics by giving the analogues of Newton's laws (see section 1).

The Newtonian postulate (1) (equivalence of inertial frames) has been replaced by the relativistic postulates (1A) and (1B).

We now add the 4-vector equivalents of postulates (2) and (3).

Postulate (2) :- Since inertial frames are equivalent, the effect of interaction, ie forces, can only be to produce an acceleration, or better a change in momentum.

So we write $$F^\mu = \frac{dp^\mu}{d\tau}$$

where τ is the proper time and $F^\mu = (F^0, \gamma \underline{f})$ is the 4-force.

So, $$F^\mu = \gamma(u) \left(\frac{d}{dt} \gamma(u) mc \, , \, \frac{d}{dt} \gamma(u) m \underline{u} \right)$$

$$= \gamma(u) \left(\frac{1}{c} \frac{dE}{dt} \, , \, \frac{d\underline{p}}{dt} \right)$$

The space cpt, $\underline{f}$, is the relativistic version of the Newtonian force vector. The extra, time, cpt. is power, ie. the rate of change of energy.

The defn. is such that $\underline{f} = \frac{d\underline{p}}{dt}$

Of course, just as in Newtonian theory, this law is really incomplete until we specify the 4-force F^μ from a particular interaction, eg. electromagnetism.

Postulate (3) :- Dynamics takes place in the arena of Minkowski spacetime. The laws of physics are therefore invariant under translations (in both $\underline{x}$ and t) and rotations / Lorentz transformations.

In particular, Noether's theorem requires:-

Invariance under spacetime translations $\Rightarrow$ conservation of 4-momentum.

The important point here is that in relativity, both 3-momentum <u>and</u> energy are conserved.

Conservation of the 4-momentum, or energy-momentum 4-vector, is the main tool we will use in analysing relativistic particle dynamics.

(Notice that it is the relativistic 3-momentum $\underline{p} = \gamma m \underline{u}$ which is conserved, <u>not</u> $m\underline{u}$)

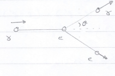
RELATIVISTIC COLLISIONS.

We now use the laws of relativistic dynamics to analyse a number of examples of particle collisions.

6.1 COMPTON EFFECT :-

This is a collision between a photon and an electron. The electron is stationary - we analyse the process in its rest frame.

This was first realised by Compton in 1922, using X-ray scattering from electrons. This was one of the first experiments to test the laws of special relativity for photons. The photons are scattered, and lose energy (increasing the X-ray wavelength.)



Total 4-momentum $= p^\mu_\gamma + p^\mu_e$

After collision, total $= p'^\mu_\gamma + p'^\mu_e$

$= p^\mu_\gamma + p^\mu_e$ by energy-momentum conservation.

Let $\quad p^\mu_\gamma = \left(\frac{1}{c} E_\gamma, \underline{p} \right) \qquad p^{\mu\prime}_\gamma = \left(\frac{1}{c} E_\gamma', \underline{p}' \right)$

where for photons $\quad E_\gamma = c|\underline{p}|$

And $\quad p^\mu_e = (mc, \underline{0}) \qquad p^{\mu\prime}_e = (\gamma mc, \gamma m \underline{v})$

Energy-momentum conservation $\Rightarrow$

$$c|\underline{p}| + mc^2 = c|\underline{p}'| + \gamma mc^2$$

$$\underline{p} = \underline{p}' + \gamma m \underline{v}$$

Rearranging, and simply writing p for $|\underline{p}|$ etc:-

$$\gamma^2 m^2 c^2 = (p - p' + mc)^2$$

$$\gamma^2 m^2 v^2 = (\underline{p} - \underline{p}') \cdot (\underline{p} - \underline{p}') = p^2 + p'^2 - 2pp' \cos\theta$$

Subtract the left hand sides:-

$$\gamma^2 m^2 (c^2 - v^2) = p^2 + p'^2 + m^2c^2 - 2pp' + 2mc(p - p') - p^2 - p'^2 + 2pp' \cos\theta$$

$$\Rightarrow \quad pp'(1 - \cos\theta) = (p - p')mc$$

$$\Rightarrow \quad 1 - \cos\theta = mc \left(\frac{1}{p'} - \frac{1}{p} \right)$$

in terms of wavelengths, since $\lambda = \frac{h}{p}$, we find

$$\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$$

The scattered photons lose energy, ie. wavelength increases, dependent on the scattering angle.

This derivation separates the conservation of energy and momentum. An equivalent derivation which exploits fully the 4-vector notation is as follows:-

$$p_\gamma \cdot p'_\gamma \equiv g_{\mu\nu}\, p^\mu_\gamma\, p^{\nu\prime}_\gamma = \frac{1}{c^2} E_\gamma E'_\gamma - \underline{p}\cdot\underline{p}'$$

$$= p p'(1-\cos\theta)$$

4-momentum conservation $\Rightarrow$ $p^{\mu\prime}_e = p^\mu_e + p^\mu_\gamma - p^{\mu\prime}_\gamma$

$$\Rightarrow \quad p'^2_e = p^2_e + p^2_\gamma + p'^2_\gamma + 2\left(p_e\cdot p_\gamma - p_e\cdot p'_\gamma - p_\gamma\cdot p'_\gamma\right)$$

$$\text{ie.} \quad -m^2c^2 = -m^2c^2 + 0 + 0 + 2\, p_e\cdot(p_\gamma - p'_\gamma) - 2\, p_\gamma\cdot p'_\gamma$$

$$\Rightarrow \quad p_\gamma\cdot p'_\gamma = p_e\cdot(p_\gamma - p'_\gamma)$$

$$\text{ie.} \quad p p'(1-\cos\theta) = mc\,(p - p')$$

as above.

Inverse Compton scattering is the related process where a photon collides with a fast (relativistic) electron. It can then gain energy in the collision.

This process is important in astrophysics as a production mechanism for inter-galactic X-rays (and energy loss for charged high energy protons).

See, eg, Rindler pp 85, 86.

6. COLLIDING BEAMS:-

Modern particle accelerators fall into 2 categories - colliding beam and fixed-target.

In a collider, 2 beams of particles are collided head-on. Examples are the $Sp\bar{p}S$ at CERN (discovery of W, Z in 1982/83), the e^+e^- collider LEP at CERN, and the ep collider HERA at DESY.

Consider a collider like LEP, colliding equal energy (and mass) particles:

$$\overset{e^-}{o} \longrightarrow \qquad \longleftarrow \overset{e^+}{o}$$

Electron 4-momentum $\quad p_1^\mu = (E/c, \underline{p})$

Positron 4-momentum $\quad p_2^\mu = (E/c, -\underline{p})$

$$p_1^2 = -E^2/c^2 + p^2 = -m^2c^2$$

$$p_2^2 = -E^2/c^2 + p^2 = -m^2c^2$$

Total 4-momentum $= p_T^\mu = (2E/c, \underline{0})$

This is in the <u>laboratory frame</u>.

The <u>centre of momentum (CM) frame</u> is defined as the frame of reference in which the total 3-momentum is zero. It is always possible to choose such a frame by making an appropriate Lorentz transformation.

Exceptionally in this case the lab and CM frames are the same.

The total energy in the CM frame is $2E$. This energy is available to create a new particle at rest with mass $m = 2E/c^2$.

In a collider like HERA, different mass particles with different energies are collided.

$$\overset{e^-}{\circ} \longrightarrow \qquad \longleftarrow \overset{p}{\circ}$$

Electron 4-momentum $\quad p_1 = (E_1/c, \mathbf{p}_1)$

Proton 4-momentum $\quad p_2 = (E_2/c, \mathbf{p}_2)$

Where $\quad p_1^2 = -E_1^2/c^2 + \mathbf{p}_1^2 = -m_1^2c^2$

$\qquad\quad p_2^2 = -E_2^2/c^2 + \mathbf{p}_2^2 = -m_2^2c^2$

Total 4-momentum $\quad p_T^\mu = \left((E_1+E_2)/c \, , \, \mathbf{p}_1 + \mathbf{p}_2 \right)$

This is in the lab. frame.

In the CM frame (denote by $'$), $\quad p_T^{\mu\,\prime} = \left((E_1'+E_2')/c \, , \, \underline{0} \right)$

The total energy available to create a new particle is $E_1' + E_2'$.

We have to determine $E_1'+E_2'$ given the energies of the 2 beams E_1 and E_2.

This can be done by explicitly finding the Lorentz transformation between the lab and CM frames. However, this is complicated and unnecessary.

The correct method is to identify Lorentz invariant quantities. i.e. are the same in all frames. In particular, the magnitude of the total 4-momentum is an invariant:-

$$p_T^2 = -(E_1+E_2)^2/c^2 + (\mathbf{p}_1+\mathbf{p}_2)^2 = -(E_1+E_2)^2/c^2 + p_1^2 + p_2^2 - 2p_1p_2$$

(since the direction of $\mathbf{p}_1$ and $\mathbf{p}_2$ is opposite).

$$= -2E_1E_2/c^2 + 2p_1p_2 - m_1^2c^2 - m_2^2c^2$$

modern colliders E_1, p_1 etc are much bigger than m_1

eg LEP has $E = 50\,\text{GeV}$; electron mass $= 0.5\,\text{MeV}/c^2$.

Note that particle masses are usually expressed in units of energy$/c^2$. Often the $/c^2$ is omitted, as it's common to use units where $c = 1$.]

Neglecting m_1, m_2, the equations simplify:-

$p_1 = E_1/c$, $p_2 = E_2/c$ (ultra-relativistic limit - same equation as for photons.)

$$\Rightarrow \quad p_T^2 = -4E_1E_2/c^2$$

But since this is invariant, $p_T'^2 = p_T^2$

$$\Rightarrow \quad E_1' + E_2' = 2\sqrt{E_1E_2}$$

The total cm energy is usually specified by giving $\sqrt{s}$. In terms of 4-momenta, the Lorentz invariant expression for s is

$$s = -\left(p_1 + p_2\right)^2$$

from which $s = E_T'^2/c^2$.

(See problem sheet 5 for HERA parameters)

6.3 FIXED TARGET ACCELERATORS.

The $Sp\bar{p}S$ accelerator at CERN accelerates protons and antiprotons to energies of 270 GeV or more before colliding them. The CM energy is $\sqrt{s}\Big|_{c=1} = 2E = 540$ GeV.

The SPS can also be used in fixed-target mode, colliding a 270 GeV proton with a proton at rest. To calculate the CM energy in this case, we follow the same method as sect 6.2.

$$P_1 = \left(E/_c , \underline{p}\right) \qquad P_2 = (Mc, \underline{0})$$

$$\Rightarrow \quad P_T = \left(\tfrac{1}{c}(E + Mc^2), \underline{p}\right)$$

$$\Rightarrow \quad P_T^2 = -\frac{1}{c^2}(E+Mc^2)^2 + p^2 \quad , \text{ where } -\frac{E^2}{c^2} + p^2 = -M^2c^2$$

$$= -2M^2c^2 - 2EM$$

$$\simeq -2EM \qquad \text{in the case } E \gg Mc^2$$

In the CM frame,

$$P_T' = \left(E_T'/_c , \underline{0}\right)$$

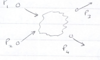
$$\Rightarrow \quad P_T'^2 = -E_T'^2/_{c^2}$$

and so the CM energy is $E_T' = \sqrt{2EMc^2}$

Notice this is much less than can be achieved in collider mode.

2-PARTICLE SCATTERING:-

Consider 2 particle → 2 particle scattering.



If the masses are preserved, ie. $m_1 = m_3$ and $m_2 = m_4$, the scattering is Elastic. (Otherwise, Inelastic.)

To describe 2 → 2 scattering, it is useful to define the following Lorentz invariants:-

$$s = -\left(P_1 + P_2\right)^2$$

$$t = -\left(P_1 - P_3\right)^2$$

$$u = -\left(P_1 - P_4\right)^2$$

The energy-momentum (ie 4-momentum) conservation laws are,

$$P_1 + P_2 = P_3 + P_4$$

The individual 4-momentum magnitudes are Lorentz invariants, ie. the masses. We have

$$P_1^2 = -m_1^2c^2, \quad P_2^2 = -m_2^2c^2, \quad P_3^2 = -m_3^2c^2, \quad P_4^2 = -m_4^2c^2.$$

Here is an important identity:-

$$s + t + u = \sum_{i=1}^{4} m_i^2 c^2$$

Proof:-

$$s + t + u = -\left(3 P_1^2 + P_2^2 + P_3^2 + P_4^2 + 2 P_1 \cdot \left(P_2 - P_3 - P_4\right)\right)$$

$$= -\left(P_1^2 + P_2^2 + P_3^2 + P_4^2\right)$$

using 4-momentum conservation to simplify final term.

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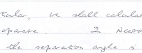
FIXED TARGET $2 \to 2$ SCATTERING

Consider $2 \to 2$ elastic, equal mass scattering with the target particle initially at rest. (This is 'relativistic billiards.')

In particular, we shall calculate the angle at which the particles separate. In Newtonian dynamics, this is $90°$. We find the separation angle is less than $90°$, i.e. the particles are emitted in a forward cone whose semi-angle shrinks as the energy is increased.

(This is very striking in fixed-target experiments at high energy, where the detectors are all in the very forward direction, in contrast to the 4π detectors at colliders.)



Lab frame

$$p_1^\mu = \left(\frac{E_1}{c}, \underline{p}_1\right) \qquad\qquad p_3^\mu = \left(\frac{E_3}{c}, \underline{p}_3\right)$$

$$p_2^\mu = \left(mc, \underline{0}\right) \qquad\qquad p_4^\mu = \left(\frac{E_4}{c}, \underline{p}_4\right)$$

4-momentum conservation $\Rightarrow$

$$\left.\begin{array}{lll} \text{(i)} & E_1 + mc^2 & = E_3 + E_4 \\ \text{(ii)} & \underline{p}_1 & = \underline{p}_3 + \underline{p}_4 \end{array}\right\}$$

$$\Rightarrow \quad p_1^2 = p_3^2 + p_4^2 + 2 p_3 p_4 \cos\theta \qquad (\theta = \theta_3 + \theta_4)$$

This is much simplified in the special case where the particles come off with the same energy, ie. $E_3 = E_4$. 3-momentum conservation then requires $p_3 = p_4$ as well, and $\theta_3 = \theta_4$.

The invariance conditions give $\quad E_1^2 = m^2c^4 + c^2p_1^2$, etc.

So we have, from (i)

$$E_1 + mc^2 = 2E_3 \quad \Rightarrow \quad E_1^2 - m^2c^4 = 4E_3\left(E_3 - mc^2\right)$$

and from (ii),

$$E_1^2 - m^2c^4 = 2\left(E_3^2 - m^2c^4\right)\left(1 + \cos\theta\right)$$

We therefore find,

$$1 + \cos\theta = \frac{2E_3}{E_3 + mc^2}$$

$$\Rightarrow \quad \cos\theta = \frac{E_3 - mc^2}{E_3 + mc^2} = \frac{E_1 - mc^2}{E_1 + 3mc^2} = \frac{K_1}{K_1 + 4mc^2}$$

We see, therefore, that for low energies where $E_1 \simeq mc^2$, $\quad \cos\theta \simeq 0 \quad$ ie $\theta \simeq 90^\circ$

However for high energies, $\cos\theta < 1$. In the limit as $E_1 \gg mc^2$, $\quad \cos\theta \simeq 1 \quad$ ie. θ small.

(K_1 is the kinetic energy, ie. total energy − rest mass energy, ie $K_1 = E_1 - mc^2$)

The analysis for different energies for the emitted particles is more difficult. It is best done in the CM frame and then boosting to the Lab frame. (See eg. Rindler, p77.)

The result is

$$\tan\theta_3 \tan\theta_4 = \frac{1}{\gamma^2(v)} = \frac{2}{\gamma(u)+1}$$

where u is the incident particle's velocity in the lab frame and v . . . CM frame.

Elastic 2→2 scattering:-

Lab frame  Assume:- Equal angle (energy) scattering.

Compare p_T^2

$$\Rightarrow (E+m)^2 - \underbrace{p_1^2}_{E^2-m^2} = (2E')^2 \qquad \text{where } E_{cm} = 2E'$$

$$\Rightarrow E' = \frac{1}{2}\sqrt{2Em+2m^2} = \frac{1}{\sqrt{2}}\sqrt{(E+m)m}.$$

To find cm frame velocity:- $E' = \gamma(v)m$

$$\Rightarrow \gamma(v) = \sqrt{\frac{E+m}{2m}}$$

$$\Rightarrow v^2 = 1 - 1/\gamma^2 = 1 - \frac{2m}{E+m} = \frac{E-m}{E+m}$$

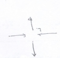
$$v = \sqrt{\frac{E-m}{E+m}}$$

OR/ $p_1' = \gamma(p_1 - Ev)$ $\quad p_2' = \gamma(p_2 - Ev)$ $\quad p_1' = -p_2'$

$$\Rightarrow p_1 - Ev = +mv$$

$$\Rightarrow v = \frac{p_1}{E+m} = \frac{\sqrt{E^2-m^2}}{E+m} = \sqrt{\frac{E-m}{E+m}}.$$

Equal angle scattering in CM frame must be:-



So all energy/mom is in transverse direction in CM, i.e.

$$E' = \gamma(u_y')\,m \qquad \text{Recall} \quad E' = \gamma(v)\,m.$$

In lab frame $\quad \tan\theta/_2 = \dfrac{u_y}{u_x} = \dfrac{u_y'}{\gamma(v)\left(1+\frac{u_x' v}{c^2}\right)}\;\dfrac{\left(1+\frac{u_x' v}{c^2}\right)}{(u_x'+v)}$

$$= \frac{1}{\gamma(v)}\,\frac{u_y'}{v}$$

Since $\quad E' = \gamma(u_y')\,m = \gamma(v)\,m \quad \Rightarrow \quad u_y' = v.$

$$\Rightarrow \tan\theta/_2 = 1/\gamma(v).$$

To compare with derivation in notes:- $\qquad \cos\theta = \dfrac{E-m}{E+3m}$

$$\Rightarrow \cos\theta = \frac{\gamma(u)-1}{\gamma(u)+3}$$

$$\cos\theta = \cos^2\theta/_2 - \sin^2\theta/_2 = \frac{\cos^2\theta/_2 - \sin^2\theta/_2}{\cos^2\theta/_2 + \sin^2\theta/_2} = \frac{1-\tan^2\theta/_2}{1+\tan^2\theta/_2}$$

$$\Rightarrow \frac{1-t^2}{1+t^2} = \frac{\gamma-1}{\gamma+3} \Rightarrow \gamma+3-t^2\gamma-3t^2 = \gamma-1+t^2\gamma-t^2$$

$$\Rightarrow \tan^2\theta/_2 = \frac{2}{\gamma(u)+1}.$$

Conversion, $\gamma(v) = \sqrt{\frac{E+m}{2m}}$ $\gamma(u) = E/m$

$$\Rightarrow \quad \frac{1}{\gamma(v)^2} = \frac{2m}{E+m} = \frac{2}{E/m + 1}$$

$$= \frac{2}{\gamma(u) + 1}$$

LORE

LORENTZ TRANSFORMATIONS of $\underline{E}$ AND $\underline{B}$:-

We will identify $\underline{E}$ and $\underline{B}$ fields by the dependence of the Lorentz force on velocity:-

$$\underline{F} = q\left(\underline{E} + \underline{u}\times\underline{B}\right)$$

where $\underline{f}$ = 3-force and $\underline{u}$ = 3-velocity of test charge q.

7.1.1 Lorentz transformation of $\underline{f}$:-

First we need the Lorentz transformation of the 3-force.

Recall that the 4-force is $F^\mu = \frac{dP^\mu}{d\tau} = \gamma(u)\frac{dP^\mu}{dt}$

where $P^\mu = \left(\gamma(u)mc,\ \gamma(u)m\underline{u}\right)$

$$\Rightarrow \quad F^\mu = \gamma(u)\left(\frac{d}{dt}\gamma(u)mc,\ \frac{d}{dt}\gamma(u)m\underline{u}\right)$$

We now show this is equivalent to

$$F^\mu = \left(\gamma(u)\,\underline{f}\cdot\underline{u}/c,\ \gamma(u)\,\underline{f}\right)$$

where the 3-force is $\underline{f} = \frac{d\underline{p}}{dt} = \frac{d}{dt}\gamma(u)m\underline{u}$.

To see this, note that $\underline{f} = \dot{\gamma} m \underline{u} + \gamma m \dot{\underline{u}}$ The

$\Rightarrow \quad \underline{f}\cdot\underline{u} = \dot{\gamma} m u^2 + \gamma m \dot{\underline{u}}\cdot\underline{u}$ and since $\dot{\gamma} = \frac{d}{dt}\left(1 - u^2/c^2\right)^{-1/2}$

$= \dot{\gamma} m c^2 \left(\frac{u^2}{c^2} + \frac{1}{\gamma^2} \right)$ $\qquad = \frac{1}{\gamma^3}\, \underline{u}\cdot\dot{\underline{u}}/c^2$

$= \dot{\gamma} m c^2$

The Lorentz transformation of F^μ are the same as for any 4-vector:-

$$F^{0\prime} = \gamma(v)\left(F^0 - \frac{v}{c} F_x\right)$$

$$F_x' = \gamma(v)\left(F_x - \frac{v}{c} F^0\right)$$

$$F_y' = F_y, \qquad F_z' = F_z$$

We need the s components,

$$\gamma(u')\, f_x' = \gamma(v)\left(\gamma(u)\, f_x - \frac{v}{c^2}\gamma(u)\, \underline{f}\cdot\underline{u}\right)$$

and using

$$\gamma(u') = \gamma(u)\,\gamma(v)\left(1 - u_x v/c^2\right)$$

$$\Rightarrow \quad f_x' = \left(f_x - \frac{v}{c^2}\underline{f}\cdot\underline{u}\right)\Big/\left(1 - u_x v/c^2\right)$$

Then, $\gamma(u')\, f'_y = \gamma(u)\, f_y$

$$\Rightarrow \quad f'_y = \frac{f_y}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)}$$

and similarly for f'_z.

Note the similarity with the velocity transformation formulae

$$u'_x = \frac{(u_x - v)}{1 - \frac{u_x v}{c^2}}$$

$$u'_y = \frac{u_y}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)} \quad , \text{ similarly for } u'_z .$$

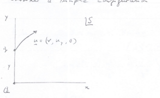
7.1.2 Electric and Magnetic Fields:-

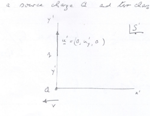
A static charge in frame S has only an electric field.

However, in frame S' (moving with velocity v along the x-direction), this will appear as a current. It will therefore have an azimuthal magnetic field.

$\Rightarrow$ Electric and magnetic fields transform into each other under Lorentz transformations!

Consider a simple configuration of a source charge Q and test charge q.





In frame S, q experiences an electrostatic force,

$$\underline{f} = (0, f_y, 0) \qquad \text{with } f_y = q\mathcal{E}, \qquad \mathcal{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{y^2}.$$

In frame S', the force on q is given by the Lorentz transformation:

$$f_x' = \frac{1}{\left(1 - \frac{u_x v}{c^2}\right)}\left(f_x - \frac{v}{c^2}\underline{f}\cdot\underline{u}\right)$$

$$= \frac{1}{\left(1 - v^2/c^2\right)}\left(f_x - \frac{v^2}{c^2}f_x - \frac{v}{c^2}u_y f_y\right) \qquad \text{and since } f_x = 0$$

$$= -\gamma(v)^2 \frac{v u_y}{c^2} q\mathcal{E} \qquad = -\gamma(v)\frac{v u_y'}{c^2} q\mathcal{E}$$

$$f_y' = \frac{1}{\gamma(v)\left(1 - v^2/c^2\right)} f_y$$

$$= \gamma(v)\, q\mathcal{E} \qquad \text{and } f_z' = 0.$$

Fitting this to the Lorentz force law $\underline{f}' = q\left(\underline{E}' + \underline{u}' \times \underline{B}'\right)$

we have to identify $\underline{E}' = \left(0, \gamma(v)\mathcal{E}, 0\right)$

$$\underline{B}' = \left(0, 0, -\gamma(v)\frac{v}{c^2}\mathcal{E}\right)$$

Recall $\underline{E} = \left(0, \mathcal{E}, 0\right)$ $\underline{B} = \left(0, 0, 0\right)$ in frame S.

By looking at a number of configurations of charges and building up to the general result from special cases, we eventually find the full transformations for $\underline{E}$ and $\underline{B}$.

These are,

$$E'_x = E_x \qquad E'_y = \gamma(v)\left(E_y - vB_z\right) \qquad E'_z = \gamma(v)\left(E_z + vB_y\right)$$

$$B'_x = B_x \qquad B'_y = \gamma(v)\left(B_y + \frac{v}{c^2}E_z\right) \qquad B'_z = \gamma(v)\left(B_z - \frac{v}{c^2}E_y\right)$$

Clearly the special case derived above agrees with the formulae for E'_y and B'_z.

7. ELECTRODYNAMICS.

It is remarkable that electrodynamics, unlike Newtonian dynamics, is already entirely consistent with special relativity, despite being formulated many years earlier.

All we need do is therefore to write the laws of electrodynamics in a form in which their special relativistic form is manifest, ie. in terms of 4-vectors, etc.

The laws of electrodynamics are summarised in Maxwell's equations (1867) (See eg. Grant & Phillips, chptrs 10, 14)

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{\epsilon_0} \rho \qquad (1) \quad \text{Gauss'}$$

$$\underline{\nabla} \times \underline{B} - \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{j} \qquad (2) \quad \text{Ampère's}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad (3) \quad \text{no mag. mono-poles}$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 \qquad (4) \quad \text{Faraday's}$$

ρ and $\underline{j}$ are the charge density and current, and satisfy the continuity equation.

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{j} = 0 .$$

The force on a charge e is described by the Lorentz force equation.

$$\underline{f} = e \left(\underline{E} + \underline{u} \times \underline{B} \right)$$

With no source, we can derive an equation describing the propagation of electromagnetic waves:-

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{E}) - \nabla^2 \underline{E} = -\nabla^2 \underline{E} \quad . \quad (1)$$

$$= -\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{B}) = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \underline{E} \quad (4,2)$$

$$\Rightarrow \quad -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \underline{E} + \nabla^2 \underline{E} = 0$$

This is the equation for propagation of electromagnetic waves with velocity $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$. There is a similar eq. for $\underline{B}$.

It is then seen that $c^2 = \frac{1}{\mu_0 \epsilon_0}$, where c = vel. of light

So we conclude that light is an electromagnetic wave.

The presence of the speed of light c in Maxwell's equations already suggests they may be compatible with special relativity as they stand.

Maxwell's eqs (3) + (4) show that the electric and magnetic fields can be written as potentials:-

$$(3) \Rightarrow \underline{B} = \underline{\nabla} \times \underline{A}$$

$$(4) \Rightarrow \underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}$$

There is some ambiguity left in this representation. This can be fixed by imposing the Lorentz gauge condition:

$$\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

With this restriction, Maxwell's eqs (1) and (2) become:-

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = -\frac{1}{\epsilon_0}\rho$$

$$-\frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} + \nabla^2 \underline{A} = -\mu_0 \underline{j}$$

These are wave equations for the potential. For time-independent potentials, the first is Gauss' law of electrostatics.

Clearly, there is a close relation in relativity between magnetic and electric fields - they must transform into each other under changes of reference frame.

It is interesting to build up the theory of electromagnetism in this way. Start from the electric field of a static charge and consider its description in a moving reference frame, consistent with relativity. See, eg, French for a discussion along these lines.

Here, we will simply demonstrate relativistic invariance by writing the above electrodynamic equations in 4-vector form.

The continuity equation becomes

$$\frac{\partial}{\partial x^\mu} J^\mu = 0$$

(usually called 'current conservation') where

$$\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla} \right)$$

and the current 4-vector is $J^\mu = (c\rho, \underline{j})$

The Maxwell eqs for the potential are

$$\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} A^\nu = -\mu_0 J^\nu$$

where the potential 4-vector is $A^\mu = \left(\frac{1}{c}\phi, \underline{A}\right)$

With the source set equal to 0, this is the standard wave equation in relativity. The Lorenz gauge is $\frac{\partial}{\partial x^\mu} A^\mu = 0$.

The electric and magnetic fields are put together into an antisymmetric tensor :-

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 \\ \frac{1}{c}E_1 & 0 & -B_3 & B_2 \\ \frac{1}{c}E_2 & B_3 & 0 & -B_1 \\ \frac{1}{c}E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

The Maxwell eqs (1) and (2) are written as

$$\frac{\partial}{\partial x^\mu} F^{\mu\nu} = \mu_0 J^\nu$$

The (3) and (4) equations become

$$\frac{\partial}{\partial x^\mu} F_{\nu\lambda} + \frac{\partial}{\partial x^\nu} F_{\lambda\mu} + \frac{\partial}{\partial x^\lambda} F_{\mu\nu} = 0$$

The relation between fields and potentials is

$$F^{\mu\nu} = \frac{\partial}{\partial x_\mu} A^\nu - \frac{\partial}{\partial x_\nu} A^\mu$$

which is in fact implied by the relativistic form of (3) and (4)

The Lorentz transformation properties of the electric and magnetic fields can be derived from first principles. They are summarised by the assignment of $\underline{E}$ and $\underline{B}$ in the field strength tensor $F^{\mu\nu}$. This has Lorentz transformation:-

$$F^{\mu\nu\,\prime} = L^\mu{}_\lambda L^\nu{}_\rho F^{\lambda\rho}$$

ie. $F' = L F L^T$ in matrix notation,

where L is the Lorentz transformation matrix of section 5.1

ie. $L^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ for S' with velocity v in x-direction.

Evaluating $F' = LFL^T$, we find the Lorentz transformations for $\underline{E}$ and $\underline{B}$ under a standard Lorentz transformation (in the x-direction).

$$E_1' = E_1 \qquad E_2' = \gamma(E_2 - vB_3) \qquad E_3' = \gamma(E_3 + vB_2)$$

$$B_1' = B_1 \qquad B_2' = \gamma\left(B_2 + \frac{v}{c^2}E_3\right) \qquad B_2' = \gamma\left(B_3 - \frac{v}{c^2}E_2\right)$$

We see therefore that relativity necessarily implies the unification of electricity and magnetism. The assignment of fields as 'electric' or 'magnetic' depends on the frame of reference.

Just as the magnitude of a 4-vector is a Lorentz invariant, we can find Lorentz invariants from the tensor $F^{\mu\nu}$. There are two:

$$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \frac{1}{c^2}\underline{E}^2 - \underline{B}^2$$

$$\frac{1}{8} \epsilon_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho} = \frac{1}{c}\underline{E}\cdot\underline{B}$$

This can be checked explicitly from the above transformations.

Finally, the Lorentz force law can be written in 4-vector form, using the field strength $F^{\mu\nu}$ and 4-velocity U^ν. The 4-force is,

$$F^\mu = -e F^{\mu\nu} U_\nu$$

Recommended Books:-

A.P. French, Special Relativity (MIT series), Chapman & Hall.

W. Rindler, Introduction to Special Relativity (Oxford)

Also,

D. Halliday, R. Resnick, J. Walker, Fundamentals of Physics (Wiley) Chapter 42.

(Hartle GR