Dynamics

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10:58

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Vectors

some physical quantities have a magnitude only - scalar

other quantities have magnitude and direction-vector quantities

Physical equations often contain both vector and scalar quantities
It is vital to distinguish between them
We will use one of the following to show that a quantity is a vector:
Bold face **a** (used in books a lot)
Underline $\underline{a}$
Overarrow $\vec{a}$
Each and every vector quantity must have one of these

2 views of a vector

1) Geometric/fundamental View
    - Vector is an **arrow in space**
        - Magnitude --> length
        - Direction --> direction of arrow
    - **Not** fixed to a start point
        - Start point is irrelevant
        - Same magnitude & direction= same vector
2) Coordinate view
    - Vector is represented by 3 numbers-
        - Giving components in 3 specified directions
        - Specified by axes
            - + choice is good
                - You get to pick the axes
                - A good choice makes the maths easier
            - - Components depend on choice of axes => not fundamental

	Geometric (fundamental)	Coordinate
		e.g. Cartesian (axes at right angles)
E.g Position 2 points	Arrow $\vec{a}$	(x,y,z)m
Adding vectors	Join arrows top to tail. $\vec{c} = \vec{a} + \vec{b}$	Add components
Multiplication by scalar (λ)	Change length of arrow by a factor λ	Change coordinates by factor λ
Differentiation		Differentiate coordinates
Integration		Integrate coordinates

		$\int \vec{a}\, dx$

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$\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ Dot/scalar

$$\vec{a} \times \vec{b} = \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

$c_i = a_i + b_i$
For i=1,2,3
Each term has a single "free" index

2 indices on δ_{ij} tell us we are dealing with the elements of a matrix
In fact this is the identity matrix with I and j labelling the rows and columns

$\varepsilon_{ijk} a_j b_k$ where

$$\varepsilon_{ijk} = \begin{cases} +1 & ijk \quad [123, 312, 231] \textit{even permutations of } 1,2,3 \\ -1 & ijk \quad [213, 321, 132] \textit{odd permutations of } 1,2,3 \\ 0 & otherwise \end{cases}$$

Whenever we swap the order of a pair of indices in ε_{ijk} we get a factor of -1
Start with (123)
Get -1 for every pair we swap
Swap an odd number of times= -1 overall (odd perms)
Swap an even number of times =1 overall (even perms)

Let's unpack this notation
$c_i = \varepsilon_{ijk} a_j b_k$
This is 3 equations for i=1,2 or 3
Let's consider i=1

$$c_1 = \varepsilon_{1jk} = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k = 9\ terms = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 \times 0(seven\ times)$$

$= +a_2 b_3 - a_3 b_2$
Ex- repeat for c2 and c3

Finally we introduce a very useful result
$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{il} \delta_{km} - \delta_{jm} \delta_{kl}$
To verify either check possible values of j,k,l,m <u>or</u> consider symmetries

e.g. swap j,k -> gives -1 from ε_{ijk} on left -2 from interchanging terms on right
If j=k, ε_{ijk} -> on left two terms cancel on right

<u>Why this is useful</u>
Consider
$\left[\vec{a} \times (\vec{b} \times \vec{c})\right]_i = \varepsilon_{ijk} a - j \left(\vec{b} * \vec{c}\right)_k$

$$\varepsilon_{ijk}\, a_j \left(\vec{b} \times \vec{c}\right)_k$$

$$-\varepsilon_{klm} b_l c_m$$

$$= \varepsilon_{ijk} \varepsilon_{klm} a_i b_l c_m$$

$$= \varepsilon_{kij} \varepsilon_{klm} a_i b_l c_m =$$

$$(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl})$$

$$a_j b_l c_m$$

$$= b_i (\vec{a} \cdot \vec{c}) - c_i (\vec{a} \cdot \vec{b})$$

$$= [\vec{a} \times (\vec{b} \times \vec{c})]_i$$

Newton's laws

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- N1: Law of inertia- A body will remain at rest or in a state of uniform motion unless acted on by a net force.
  - Important points
    - Often obscured by friction
      - Things typically slow down as there is almost always some frictional/drag force acting
    - v=constant means that an object's vector has constant length and direction
      - Relative to specified axes/frame of reference
      - If our axes accelerate or rotate, the components of v relative to these axes will change
  - Work in inertial frames whenever possible. Remember the earth rotates (on large scales).
- N2: F=ma
  - First we need the concept of momentum.
    - $\vec{P} = m\vec{v}$
- N3: Every action has an equal and opposite reaction
  - When two bodies interact, they exert equal and opposite forces on one another
  - Doesn't require things to be at rest or in contact
- If no external forces are applied to a system, the total of the system is conserved

N2:

$$\frac{d}{dt}\bar{p} = \underbrace{\sum \bar{F}_{ext}}_{sum\ of\ forces\ on\ particle}$$

If m=constant,

$$\frac{d}{dt}\bar{p} = \frac{d}{dt}(mv) = \left(\frac{d}{dt}m\right)\bar{v} + m\left(\frac{d}{dt}\bar{v}\right) = m\left(\frac{d}{dt}\bar{v}\right)$$

In this case

$$\frac{d\bar{p}}{dt} = m\frac{d\bar{v}}{dt} = \sum \bar{F}$$

Or

$$\bar{F} = m\bar{a}$$

Forces

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We think there are 4 fundamental forces

<u>Gravity</u>
Long range force
Given two masses m and m with relative position $r_{1,2}$
Newtons law of gravitation

$$f_{12} = \frac{Gm_1m_2}{|r_{12}|^2} r_{12}$$

Using

$$r_{12} = |r_{12}| \times r_{12}$$

$$f_{12} = \frac{Gm_1m_2r_{12}}{|r_{12}|^3}$$

This is an approximation to general relativity- corrections are small unless objects are v. big / v. dense/v. fast

$$f_{12} = Gm_1m_2 \frac{r_{12}}{|r_{12}|^3}$$

Works for point objects

<u>Gravity just above the earth's surface</u>

To find out the gravitational forces of the whole earth, we need to add the contributions of every lump of rock etc. in the earth.
In general, this would be a hard calculation, but if we take the earth to be spherical it simplifies greatly
For any spherical mass distribution the gravitational force on a test object is the same as if all the mass closer to the centre than the test object was collected a the centre.

At the surface of the earth, we can take all the mass of the earth to be concentrated at the centre

For the earth,
$Mass = 5.98 \times 10^{24} kg$
$mean\ radius = 6.37 \times 10^6 m$
$G = 6.67 \times 10^{-11} Nm^2kg^{-2}$
Close to the earth's surface, $|r_{12}|$ is very close to the radius of the earth, so we can write $f = m_2 g$, where g is directed towards the centre of the earth. And
$|g| = \frac{Gm_1m_2}{|r_{12}|^2} = 9.81m^{-2}$
If we move away from the surface, the distance to the centre increases and the gravitational force drops.
<u>Electromagnitism</u>- long range
For a particle of charge e and velocity <u>E</u> and magnetic field <u>B</u>, the lorentz force <u>f</u>=e(<u>E</u>+<u>v</u><u>B</u>)*
For a point charge at teh origin,

$$E(r) = \frac{qr}{4\pi\varepsilon_0 |r_2|^2}$$

<u>Weak force</u>: Short range
B decay

<u>Strong force</u>: short range
Holds quarks together to make p & n

<u>Effective forces</u>

Most of our examples will involve various manifestations of these forces
Rather than work with the underlying fundamental forces, we just consider their cumulative effect

- E.g.
- Tension in a string
    - If we pull on one end of a string, the force is transmitted to the other end. The force is parallel to the string. We typically call this tension.
    - The string may stretch- often consider an idealised inextensible string
    - If we need to consider the stretching of the string, we use Hooke's law: for small extensions, the extension is proportional to the tension. (also good approximation for springs)
- Friction

Complicated origin-
- depends on materials
- Surface roughness
- Etc.

Frictional forces do what they can to prevent motion. This means the frictional force "tunes" itself to prevent motion if it can.

Approximation
- The maximum frictional force is given by (the coefficient of friction)*(the magnitude of the force pushing surfaces together) I

a car of mass 10^3 kg
brakes uniformly to rest from
a speed of $20\ ms^{-1}$

$F = ma$

$a = {}^{\Delta d}/_{\Delta t}$

$= {}^{20}/_{2}$

$= 10\ ms^{-1}$

$F = 10^3 \times 10$

$= 10^4\ kg\,ms^{-1}$

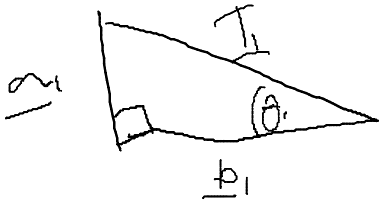
$= 10^4\ N$

Where does the force come from
friction between tyres & road

a post has 2 strings
attached find resultant force

$\sqrt{25} = 5N$

Stroud F8 frame 13
p256



$|\underline{a}_1| = T_1 \sin\theta_1$

No net force

$T_1 = \frac{\sin\theta_1}{|\underline{a}_1|}$

$T_2 \sin\theta_2 + T_1 \sin\theta_2 = 10$

$T_1 \cos\theta_1 = T_2 \cos\theta_2$

$T_1 = \frac{\cos\theta_2}{\cos\theta_1} T_2$

$T_2 = \frac{\cos\theta_1}{\cos\theta_2} T_1$

$T_2 \left[\frac{\cos\theta_2}{\cos\theta_1} \sin\theta_1 + \sin\theta_2 \right] = 10g$

$T_2 = \frac{10g \cos\theta_1}{\cos\theta_2 \sin\theta_1 + \sin\theta_2 \cos\theta_1}$

$T_1 = \frac{10g \cos\theta_2}{\cos\theta_2 \sin\theta_1 + \cos\theta_1 \sin\theta_2}$

(4) Person in a Lift

④ Person in a lift

a person stands on a set of scales in a stationary lift, the scales read m - mass of the person

Force applied to the scales is mg

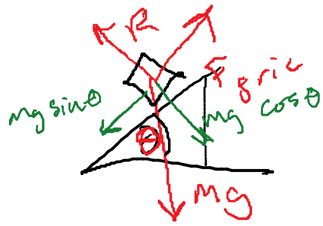
$$F = ma + mg$$

$$F/g = m\left(1 + a/g\right)$$

⑤ Friction

Block of mass m on a rough inclined plane (angle θ to horizontal)

block & plane have coefficient of friction μ



mg sinθ

mg cosθ

$mg\sin\theta = F_{fric}$

$F_{fric} = \mu R$

$R = mg\cos\theta$

$\sin\theta = mg\cos\theta\,\mu$

$R = mg$

$mg \sin\theta = mg \cos\theta\, N$

$\frac{\sin\theta}{\cos\theta} = N$

$\tan\theta = N$

$\theta = \tan^{-1}$

Two common applications of newton

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12:06

Constant acceleration

Notation

$$\ddot{\underline{x}} = \frac{d^2 x}{dt^2}$$

$$\dot{\underline{x}} = \frac{dx}{dT}$$

If

$\ddot{\underline{x}}$ = constant = $\underline{a}_c$

c indicates that this is constant

we can intergrate easily

integrating once

$$\underline{V} = \dot{\underline{x}} = \underline{V}_0 + a_c T$$

vector constant of integration

Integrate again

$$\underline{x} = \underline{x}_0 + \underline{V}_0 t + \frac{1}{2} \underline{a}_c t^2$$

another vector integration constant

Projectiles

A cannon ball is fired at speed v and angel theta to the horizontal. If the ground is flat and air resistance negligible, find where it lands
1st specify axes.: let cannon ball start at origin with its initial velocity in the (x,z) plane (with z pointing upwards)

This gives:

$$\underline{r}(t=0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{v}(t=0) = \frac{dr}{dt}\Big|_{t_0}$$

$$= \begin{pmatrix} v\cos\theta \\ 0 \\ v\sin\theta \end{pmatrix}$$

$$\underline{a} = \frac{d^2 r}{dt^2} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

$$\underline{v} = \int a \, dt = \begin{pmatrix} 0 \\ 0 \\ gt \end{pmatrix} + \underline{v_0}$$

Fix this by looking at $t=0$

when we know

$$\underline{v}(t=0) = \begin{pmatrix} v\cos\theta \\ 0 \\ v\sin\theta \end{pmatrix}$$

$$\Downarrow$$

$$\underline{v} = \begin{pmatrix} v\cos\theta \\ 0 \\ v\sin\theta - gt \end{pmatrix}$$

$|v\sin\theta - g\ldots$

similarly,

$$\underline{r} = \int \underline{v}\,dt = \begin{pmatrix} v\cos\theta\, t \\ 0 \\ v\sin\theta\, t - \frac{g}{2}t^2 \end{pmatrix}$$

at $t = 0$ we know

$$\underline{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad \underline{r} = \begin{pmatrix} v\cos\theta\, t \\ 0 \\ v\sin\theta\, t - \frac{g}{2}t^2 \end{pmatrix}$$

landing at $z = 0 \Rightarrow v\sin\theta\, t - \frac{1}{2}g t^2 = 0$

$t = 0$ is launch

$$v\sin\theta - \frac{gt}{2} = 0 \quad \text{or} \quad t = \frac{2v\sin\theta}{g}$$

is landing time

$$\underline{r}_{\text{landing}} = \begin{pmatrix} v\cos\theta\, \frac{2v\sin\theta}{g} \\ 0 \\ 0 \end{pmatrix}$$

$$\left(2v^2 \frac{\sin\theta\cos\theta}{\ldots} \right) \qquad \left(\frac{v^2}{g} \sin 2\theta \right)$$

$$= \begin{pmatrix} 2v^2 \dfrac{\sin\theta\cos\theta}{g} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{v^2}{g} \sin 2\theta \\ 0 \\ 0 \end{pmatrix}$$

Comments

motion in x, y, z directions

is independent

we have familiar $v = u + at$

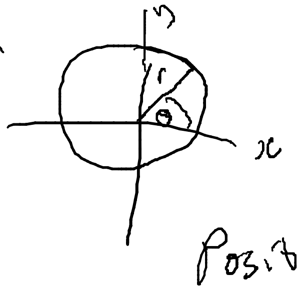
$$s = ut + \tfrac{1}{2} at^2$$

$$v^2 = u^2 + 2as$$

results for constant acceleration

Circular motion

Consider a particle moving in a circle of radius r centered at origin



Position $\underline{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

angle θ depends on time

for uniform motion, θ grows linearly with time

$$\theta = \theta_0 + \omega t$$

ω = angular speed / frequency (units $\frac{\text{radians}}{\text{second}}$)

what is the particles acceleration?

we know $\underline{r} = \begin{pmatrix} r\cos(\theta_0 + \omega t) \\ r\sin(\theta_0 + \omega t) \end{pmatrix}$

$\Downarrow$

$$\underline{V} = \frac{dr}{dt} = \dot{\underline{r}} = \begin{pmatrix} -r\omega\sin(\theta_0 + \omega t) \\ r\omega\cos(\theta_0 + \omega t) \end{pmatrix}$$

$$\boxed{\frac{d}{dt} f \circ g(t) = \frac{df}{dg}\frac{dg}{dt}}$$

Strand P7

f1, f5-8

$$\underline{a} = \frac{d^2 \underline{r}}{dt^2} = \frac{dv}{dt} = \begin{pmatrix} -r\omega^2\cos(\theta_0 + \omega t) \\ -r\omega^2\sin(\theta_0 + \omega t) \end{pmatrix}$$

$$= -r\omega^2 \begin{pmatrix} \cos(\theta_0 + \omega t) \\ \sin(\theta_0 + \omega t) \end{pmatrix}$$

Centripedal acceleration

Force required

$$F = ma = -mr\omega^2 \begin{pmatrix} \cos(\theta_0 + \omega t) \\ \sin(\theta_0 + \omega t) \end{pmatrix}$$

Force required to produce centripedal acceleration

$$|\vec{\underline{F}}| = mr\omega^2$$

and $\underline{f}$ is directed towards center of circle

<u>N.B.</u> $|\underline{v}| = r\omega$ or $v = r\omega$

then $f = mr\omega^2$

$= mv\omega$

$= \underline{mv^2}$

$$= \frac{mv^2}{r}$$

$$F = \frac{mv^2}{r}$$

$$F = ma$$

$$ma = \frac{mv^2}{r}$$

$$a = \frac{mv^2}{r}$$

$$\sqrt{\frac{ar}{m}} = v$$

Work, power and energy

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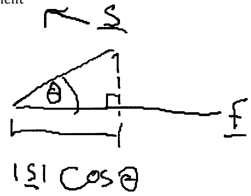
A force does <u>work</u> when it moves its point of application: work done is the magnitude of the force * the distance moved in the direction of the force

displacement



If the displacement is $\underline{s}$, the distance moved in the direction of the force is: $|\underline{s}| \cos\theta$

work done:

$$|\underline{f}|\,|\underline{s}| \cos\theta = \underline{f}.\underline{s}$$

Unit=Joule: $1J \equiv 1Nm$

Power

Power is the ratio of doing work. If force $\underline{f}$ moves its point of application by $\delta\underline{s}$ in time δt. The work done $\delta w = \underline{f}.\delta\underline{s}$

Power $P = \frac{\delta w}{\delta t} = \underline{f}.\frac{\delta \underline{s}}{\delta t} \underset{small}{\overset{\delta t}{\rightarrow}} \underline{F}.\underline{v}$

(at any instant)

Unit=watt $1w = 1\frac{J}{s}$

<u>Energy</u>
Energy is the capacity to do work. Energy comes in many forms: heat, kinetic, gravitational potential, chemical

Energy can be converted from one form to another but can't be created or destroyed

E.g.
<u>Kinetic energy</u>
A mass m, initially at rest is acted upon by a constant force $\underline{f}$

$N_2 \rightarrow \underline{F} = m\underline{a} \rightarrow \underline{a} = \frac{F}{m}$
As $\underline{F}$ is constant,

$\underline{v} = \int \underline{a}\, dt = \frac{Ft}{m} + \underline{v_0}$

$\underline{v_0}$ is 0 by initial conditions

Integrate again
$\underline{r} = \int \underline{v} dt = \underline{F}\frac{t^2}{2m} + r_0$

take origin of coords to be at the starting point of the mass $=> r_0 = 0$

$$W.D. = \underline{F}\cdot\underline{r} = \underline{F}\cdot\left(\frac{\underline{F}\, t^2}{2m}\right)$$
$$= \frac{m}{2}\left(\frac{T\underline{F}}{m}\right)\cdot\left(\frac{T\underline{F}}{m}\right)$$
$$= \frac{m}{2}\,\underline{v}\cdot\underline{v}$$

$$KE = \tfrac{1}{2}mv^2$$

Potential energy
Mass m raised from rest to rest through height h

Valid only if g is constant

Force required -mg
Work done

$$-m\underline{g}\cdot h\underline{\hat{z}} = m|\underline{g}|h$$

↑ upward unit vector

Work done by force converted to potential energy

Conservation laws
We have already mentioned conservation of momentum and energy
Conservation of energy is not always manifest
    There are many forms f energy and energy can convert from one form to another
When energy is converted to head, it is effectively "lost" to our dynamical system

Consider dragging an object over a rough surface. The energy expended in getting the object from a to b depends on the path taken
The frictional force is "non-conservative"

Other forces are conservative. A force is conservative if
1. The work done in moving from a to be is independent of path taken
   Equivalently
2. We can associate a potential energy with the state of the system (here. The position)

E.g. gravitational potential of objects mas m_1 at r_1 and m_2 at r_2

$$\text{Potential } V(\underline{r}_1,\underline{r}_2) = \left.\frac{-Gm_1m_2}{|\underline{r}_1-\underline{r}_2|}\right\}$$

Scalar quantity depends on $|\underline{r}_1-\underline{r}_2|$

Recap
Friction "steals" your energy => don't get it back
Conservative forces "give you your energy back"
    More precisely, the work done in moving from a to b is independent of path
Also we can associate a potential energy with the state of the system
    More on maths underlying this next year

Forces and Potentials
The potential encodes all the information about a conservative force
$\underline{F} = (-\frac{\delta v}{\delta x}, -\frac{\delta v}{\delta y}, -\frac{\delta v}{\delta z})$
Where d/dn means differentiate WRT n (n=x,,y,z) treating other two as constant
Partial derivatives => they are easy

$$V = -\frac{Gm_1m_2}{|\underline{r}_1-\underline{r}_2|}$$

$$= \frac{-Gm_1m_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$$

For the force on particle we need to do our partial derivatives with x, y and z (in each case, x_2 y_2 and z_2 anre constant)

$$\frac{\partial V}{\partial x_1} = \frac{\partial}{\partial x_1}\frac{-Gm_1m_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$$

this is the only thing that is hit by the partial derivative $\frac{\partial}{\partial x}$

$$= \frac{-Gm_1m_2\left(-\frac{1}{2}\right).2\,(x_1-x_2)}{\left((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\right)^{3/2}}$$

$$= \frac{Gm_1m_2(x_1-x_2)}{|\underline{r}_1-\underline{r}_2|^3}$$

repeat for $\frac{\partial V}{\partial y}$, $\frac{\partial V}{\partial z}$

$$F_{12} = -\left(\frac{\partial V}{\partial n}\right) \quad n = x_1, y_1, z_1$$

$$= -\frac{Gm_1m_2}{|\underline{r}_1-\underline{r}_2|^3}\begin{pmatrix} x_1-x_2 \\ y_1-y_2 \\ z_1-z_2 \end{pmatrix}$$

$$\frac{Gm_1m_2}{|\underline{r}_1-\underline{r}_2|^3}(\underline{r}_2-\underline{r}_1)$$

Energy conversion

For a system of particles with conservative forces acting between them the sum of the KE and PE is conserved

e.g $\frac{d}{dt}(ke + Pe) = 0$

Eg 2 particles $KE = \frac{m_1}{2}|\underline{V}_1|^2 + \frac{m_2}{2}|\underline{V}_2|^2$

$V = V(\underline{r}_1 - r_2)$

Therefore

$$\frac{d}{dT}(ke + Pe) = \frac{d}{dT}\left(\frac{m_1}{2}|\underline{V}_1|^2 + \frac{m_2}{2}|\underline{V}_2|^2 + V(r_1 - r_2)\right.$$

$$= \frac{m_1}{2} 2 V_1 \cdot \frac{d v_1}{dT} + \frac{m_2}{2} 2 r_2 \frac{d \underline{V}_2}{dT} + \frac{dV}{dT}$$

V depends on S or $\underline{r}_1$ & $\underline{r}_2$ in a small time interval ΔT, the change in V is given by

$$\Delta V = \frac{\partial V}{\partial x_1}\Delta x_1 + \frac{\partial V}{\partial y_1}\Delta y_1 + \frac{\partial V}{\partial z_1}\Delta z_1$$

$$+ \frac{\partial V}{\partial x_2}\Delta x_2 + \frac{\partial V}{\partial y_2}\Delta y_2 + \frac{\partial V}{\partial z_2}\Delta z_2$$

Then $\frac{\partial V}{\partial x_1}\frac{\Delta x_1}{\Delta T} + \frac{\partial V}{\partial y_1}\frac{\Delta y_1}{\Delta T} + \frac{\partial V}{\partial z_1}\frac{\Delta z_1}{\Delta T}$

$$\frac{\partial V}{\partial x_2}\frac{\cancel{\Delta x_2}}{\Delta t} + \frac{\partial V}{\partial y_2}\frac{\Delta y_2}{\Delta t} \quad \frac{\partial V}{\partial z_2}\frac{\Delta z_2}{\Delta t}$$

as $\Delta t \to 0$ $\quad \frac{\Delta V}{\Delta t} \to \frac{dV}{dt}$

$$\frac{\Delta x_1}{\Delta t} = \frac{dx_1}{dt} = (\underline{V}_1)_x$$

recall $\frac{dV}{dx} = (-\underline{f}_{12})_x$ etc.

as $\Delta t \to 0$ $(*) \to$

$$\frac{dV}{dt} \sum_{x=x}^{z} (\underline{V}_1)_x (-\underline{f}_{12})_x$$

$$+ \sum_{x=x}^{z} (\underline{V}_2)_x (-\underline{f}_{12})_x$$

$$= -\underline{V}_1 \cdot \underline{f}_{12} - \underline{V}_2 \cdot \underline{f}_{12}$$

$$\frac{d}{dt}(ke + pe) = m_1 \underline{V}_1 \cdot \underline{a}_1 + m_2 \underline{V}_2 \cdot \underline{a}_2 - V_1 \cdot \underline{f}_{12} - \underline{V}_2 \cdot \underline{f}_{21}$$

$$= \underline{V}_1 \cdot (m_1 \underline{a}_1 - \underline{f}_{12}) + \underline{V}_2 \cdot (m_2 \underline{a}_2 - \underline{f}_{21})$$

$= 0$ by N_2

$$\frac{d}{dt}(ke + pe) = 0$$

$$\frac{d}{d\tau}(Ke + Pe) = 0$$

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$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

eg $f = x^2$

$$f(x + \Delta x) = (x + \Delta x)^2$$

$$= x^2 + 2x\Delta x + \Delta x^2$$

$$\lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f x}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

Conservation	Invarience under
Momentum	Spatial translations
Energy	Time translations
Angular momentum	rotations

Collisions

If we neglect the energy losses during the collision, we say the collision is elastic. In this case, there is no long range potential-> we only have kinetic energy

For elastic collisions, kinetic energy is conserved.
e.g. collision of 2 snooker balls

Momentum conservation

$$m_1 \underset{\sim}{u}_1 + m_2 \underset{\sim}{u}_2 = m_1 \underset{\sim}{v}_1 + m_2 \underset{\sim}{v}_2 \quad (1)$$

Energy conservation

$$\tfrac{1}{2} m_1 \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1 + \tfrac{1}{2} m_2 \underset{\sim}{u}_2 \cdot \underset{\sim}{u}_2 =$$

$$\tfrac{1}{2} m_1 \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1 + \tfrac{1}{2} m_2 \underset{\sim}{v}_2 \cdot \underset{\sim}{v}_2 \quad (2)$$

If $m_1 = m_2$, $\underset{\sim}{u}_2 = 0$

$$(1) \Rightarrow \quad \underset{\sim}{u}_1 = \underset{\sim}{v}_1 + \underset{\sim}{v}_2 \quad (3)$$

$$(2) \quad \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1 = \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1 + \underset{\sim}{v}_2 \cdot \underset{\sim}{v}_2 \quad (4)$$

$$(3) \cdot (3) \quad \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1 = (\underset{\sim}{v}_1 + \underset{\sim}{v}_2) \cdot (\underset{\sim}{v}_1 + \underset{\sim}{v}_2)$$

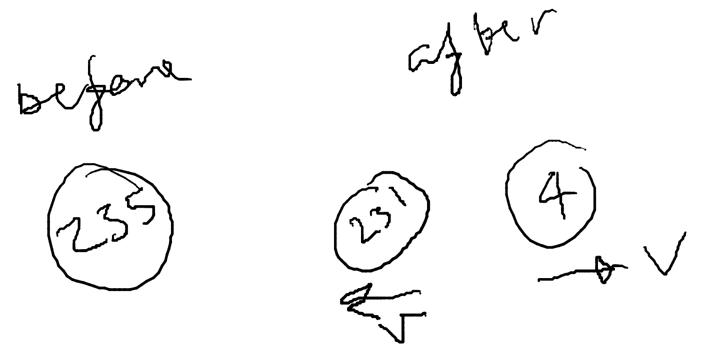
$$= \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1 + 2 \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 + \underset{\sim}{v}_2 \cdot \underset{\sim}{v}_2 \quad (5)$$

$$(5) - (4) \Rightarrow \quad 0 = 2 v_1 \cdot \underset{\sim}{v}_2$$

$\underline{V}_1$ and $\underline{V}_2$ are either perpendicular or one is zero

(2) U^{235} decays to $Th^{231} + He^4$

α has KE 4.60 MeV. If U^{235} initially at rest, find the KE of the recoiling Th nucleus



momentum conservation

KE not conserved

$$0 = 231\,v - 4V$$

$\Downarrow$

$4\underline{V}$

.... ??

$$v = \frac{4V}{231}$$

$$KE_{Th} = \frac{1}{2} m_{Th} v^2 = \frac{1}{2} m_{Th} \left(\frac{4V}{231}\right)^2$$

$$= \frac{1}{2} \frac{m_{Th}}{m_\alpha} \left(\frac{4}{231}\right)^2 m_\alpha V^2$$

$$= \frac{m_{Th}}{m_\alpha} \left(\frac{4}{231}\right)^2 \left(\frac{1}{2} m_\alpha V^2\right)$$

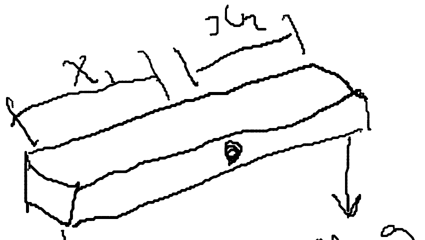
$$= \frac{231}{4} \left(\frac{4}{231}\right)^2 KE_\alpha$$

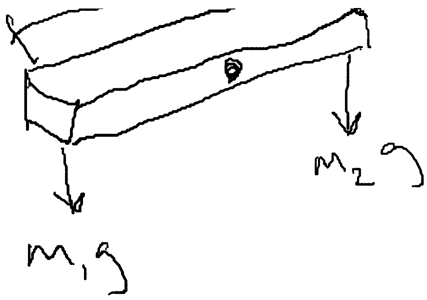
$$= \frac{4}{231} \times 4.60 = 8.0 \times 10^{-2}\ \text{MeV}$$

most energy goes into light particles

Solid bodies
Consider a set of point masses held rigidly together to form a single lump. Ignore internal forces
For extended objects we can apply forces at different places and we must take this into account

Momentum/Torque
Consider a bar on a pivot with masses hanging from either side

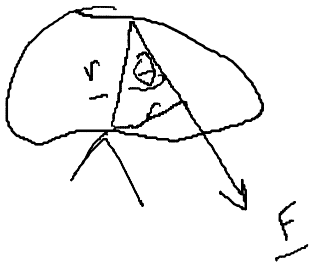




The suspended masses try to rotate the bar
The turning effect or moment of the force is determined by the force and the perpendicular distance of its line of action from the point.

$$|\text{moment}| = |\text{force}| \times \left(\text{perpendicular distance of line of action of force from pivot}\right)$$

in our simple bar example, we can think of clockwise and anticlockwise moments which must balance for equilibrium
i.e. $x_2 m_2 g = x_1 m_1 g$
In a more general situation



$\underline{f}$ acts at a point with position $\underline{r}$ relative to pivot

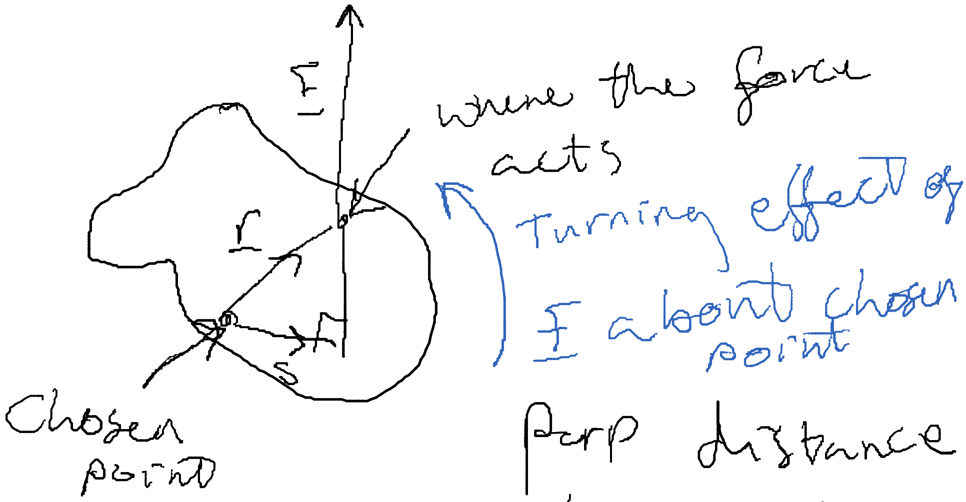
The perp distance is $|\underline{r}| \sin\theta$

$$\Rightarrow |\text{moment}| = |\underline{f}||\underline{r}| \sin\theta$$

Solid bodies

29 October 2010
10:05

We need to take account of where the forces are applied
We consider themoment of the force around some point (typically a pivot/hinge. We get to pick this)



Perp distance to line of action of force

$$|\underline{\text{moment}}| = |\underline{F}|\, s$$

Using trig $s = |\underline{r}| \sin\theta$

$$= |\underline{F} \times \underline{r}|$$

When we can associate a direction with the moment using the cross product

The moment is $\underline{M} = \underline{r} \times \underline{F}$

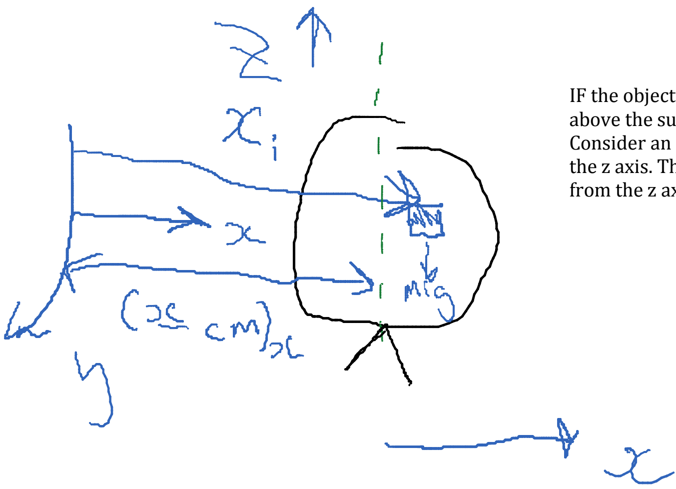
In our diagram, the force would turn the object anti clockwise about the chosen point. Using a right hand rule with fingers along the direction of rotation, our thumb points along $\underline{m}$
In rotational dynamics moment plays the same role as forces in linear dynamics
If we want an object to remain at rest without rotation we need: no net forces acting (N1)
No net moment acting (rotational analogue of N1)

<u>Centre of mass</u>
Consider a body made up of a set of point masses m at positions $\underline{x}$:we define the centre of mass to be at

$$\underline{x}_{cm} = \frac{\sum_i m_i \underline{x}_i}{\_}$$

$$\sum_i m_i$$

This is the point under which a support should be placed for the object to balance
To see that these two definitions agree, consider an object on a support



IF the object is balanced, the centre of mass is directly above the support
Consider an element of the object at distance x_i from the z axis. The centre of mass is a distance (x cm)_x from the z axis

Now consider the moment of the weight of this element about the pivot.
As drawn,, this provides a clockwise moment m_i *g(x_i *(x cm)_x)

Summing over the moments of all the elements we should get zero as the object is balanced

$$\Downarrow$$

$$\sum m_i g\left(x_i - (\underline{x}_{cm})_x\right) = 0$$

$$\Downarrow$$

$$\sum m_i x_i = \sum m_i (\underline{x}_{cm})_x$$

$$\Downarrow$$

$$(\underline{x}_{cm})_x = \frac{\sum m_i x_i}{\sum m_i}$$

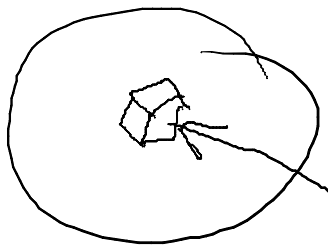
Similarly for the y and z

Components of $\underline{x}$ cm

For a set of point masses

$$\underline{X}_{cm} = \frac{\sum m_i \underline{x}_i}{\sum m_i}$$

For a continuous body we replace our point masses with a small volume element



element of volume dV with density ρ

in general ρ is a function of position

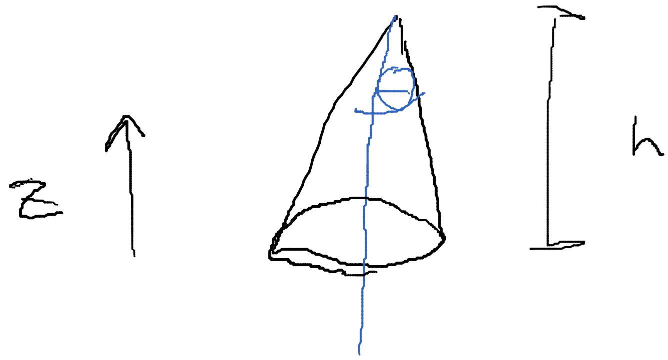
the mass of the element is $\rho\, dV$ m_i is replaced by $\rho\, dV$ and $\sum_i$ is replaced by $\int_{\text{volume}}$

$$\frac{\int dV \rho \underline{x}}{\int dV \rho}$$

$$\int dV\, \rho$$

E.g uniform core

$$\rho = \text{constant}$$



By symetry, the centre of mass lies along the centre line=> we only need to find its height
Take the base to be in the z=0 plane
We can find the z component of the centre of mass using

$$z_{cm} \quad \frac{\int dV\, \rho\, z}{\int dV\, \rho}$$

To do the integration, consider the cone to be made of disks of thickness dZ

A disk at height z is at h-z below apex
By trig, the radius of this disk is $(h-z)\tan\theta$

The volume of the disk is

$$dz\, \pi (h-z)^2 \tan^2\theta$$

So volume $dV = \pi (h-z)^2 \tan^2\theta \, dz$

$$Z_{cm} = \frac{\int \cancel{\pi}(h-z)^2 \cancel{\tan^2\theta}\, dz\, z}{\int \cancel{\pi}(h-z)^2 \cancel{\tan^2\theta}\, dz}$$

$$= \frac{\left(\frac{h^2 z^2}{2} - \frac{2hz^3}{3} + \frac{z^4}{4}\right)_0^h}{\left(h^2 z - 2h\frac{z^2}{2} + z^3/3\right)_0^h} = \frac{h}{4}$$

Solid bodies

04 November 2010
08:58

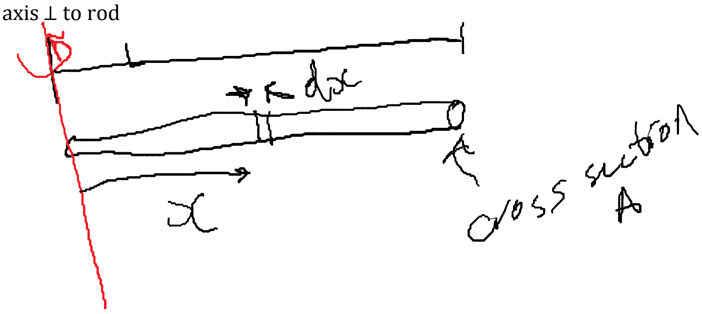
IF a force F acts at a position r relative to some point, it generates a moment about that point given by $M = r \times F$

Centre of mass
$x_{cm} = \frac{\sum_i m_i x_i}{\sum_i m_i} \equiv \frac{\int dV \rho(x) x}{\int dV \rho(x)}$

In general, $\rho(x)$ depends on x

Angular velocity
Consider a body rotating around an axes. If the object rotates at a rate of ω radians per second, the angular velocity ω is a vector with $|\omega| = \omega$ (angular speed), parallel to the rotation axis using
right hand rule

Velocity and angular velocity
Consider an element of the body t position r
Call the perpendicular distance of this element from the rotational axis d
In a small time interval, δt the object rotates through an angle $\Delta\theta = \omega\Delta t$
The element moves a distance $d\Delta\theta = d\omega\Delta t$
The speed of the element is $\frac{d\omega\Delta t}{\Delta t} = d\omega$
The displacement is perpendicular to ω and r (into the board)
The displacement is parallel to $\omega \times r$
Now $|\omega \times r| = |\omega||r|sin\phi = |\omega|d =$ speed of element
The velocity of the element $V = \omega \times r$
Rotational KE
We can find the total kinetic energy of our lump by adding up the KE of all the elements.
Consider the element previously at perpendicular distance d from the axis. Recall that its speed is dω. If the density is ρ and the element has a volume dV, mass of element is ρdV
The kinetic energy of this element is $\frac{1}{2}(\rho dV)(d\omega)^2 = \left(\frac{1}{2}mv^2\right)$
Summing (integrating) over all the elements, we find the rotational KE of the whole lump
$KE_{rot} = \int dv \frac{\rho}{2} d^2\omega^2$
Omega is a constant for the whole object(rho isn't)
$KE_{rot} = \frac{\omega^2}{2}\int dv\rho d^2$ --> moment of inertia, I
$= \frac{1}{2}I\omega^2$
I depends on the position and orientation of the rotational axis and the distribution of mass within the object
Examples
(1) thin uniform rod about end, axis ⊥ to rod



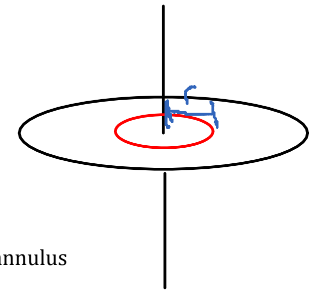
Slice the rod into short sections of length dx. Volume of element dV=Adx
The rod is uniform =>ρ=constant
$$I = \int dV\rho d^2 \rightarrow \int Adx\rho x^2 = A\rho\int_0^L dx\, x^2 = A\rho\left|\frac{x^3}{3}\right|_0^L = \frac{A}{3}\rho L^3 = \frac{(AL \times \rho)L^2}{3}$$

Angular velocity

For a point at position r relative to an origin on the rotational axis, v=ω*r
The rotational kinetic energy of the object
$Ke_{rot} = \frac{1}{2}(\int dv\rho d^2)\omega^2$
The d is the perpendicular distance of an element from the rotational axis

$Ke_{rot} = \frac{1}{2}(\int dv\rho d^2)\omega^2$

The d is the perpendicular distance of an element from the rotational axis



Example 2

Uniform disk rotating about an axis through its centre perpendicular to disk

Recall $I = \int dV\rho d^2 \rightarrow \rho \int dV d^2\ as\ \rho$ is constant in this case

Consider annuli, radians r, thickness dr

The circumference of the annulus 2pi r. If the height of the disk is t, the volume of the annulus is $(2\pi r)dr\ t$

The mass of the annulus $(2\pi r)t\rho dr$

All of the annulus is at distance from the axis

The moment of inertia of the annulus is $2\pi r t\rho dr^2$

The moment of intertia of the whole disk

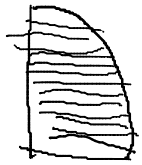
$$I_{disk} = \int_0^R (2\pi t\rho)r^3 dr = 2\pi t\rho \left|\frac{r^4}{4}\right|_0^R = \frac{(\pi t R^2)\rho R^2}{2} = \frac{mR^2}{2}$$

Where M is the mass of the disk and R is its radius

Exercise

Determine the moment of inertia of a uniform sphere (mas m, radius R) rotating about an axis through centre

Hint: Consider the sphere as a set of thin disks



Angular momentum

05 November 2010
10:25

For a point particle with mass m, velocity v and position r It's angular momentum is given by $\bar{L} = m\bar{r} \times \bar{v}$
For a set of point particles $\bar{L}_{total} = \sum_{particles} m_i \bar{r}_i \times \bar{v}_i$
For a collection of particles interacting via pairwise forces directed between the particles (i.e. no external forces acting) then $\bar{L}_{total}$ is conserved. i.e. $\frac{d}{dt}\bar{L}_{total} = 0$
Proof
$\bar{L}_{total} = \sum m_i \bar{r}_i \times \bar{v}_i$
$\frac{d}{dt}\bar{L}_{total} = \sum m_i [\bar{r}_i \times \bar{a}_i + \bar{v}_i \times \bar{v}_i] = \sum[\bar{r}_i \times (m_i \bar{a}_i)] = \sum \bar{r}_i \times \bar{F}_i$
$\bar{F}_i$ is the sum of the forces on particle i due to all other particles

$$\bar{F}_i = \sum_{j \neq i}{}^{'} \bar{F}_{ij}$$

$$\frac{d}{dt}\bar{L}_{tot} = \sum_i \sum_{j \neq i}{}^{'} \bar{r}_i \times \bar{F}_{ij}$$

Expand the summations

$$\frac{d}{dt}\bar{L}_{tot} = \begin{pmatrix} 0 & \bar{r}_1\bar{F}_{12} & \bar{r}_1\bar{F}_{13} & \bar{r}_1\bar{F}_{1n} \\ \square & 0 & \square & \bar{r}_2\bar{F}_{2n} \\ \bar{r}_3\bar{F}_{31} & \square & 0 & \square \\ \square & \square & \square & 0 \end{pmatrix}$$

We can pair up the terms, each pair is of the form $\bar{r}_1 \times \bar{F}_{12} + \bar{r}_2 \times \bar{F}_{21}$
By N3 $\bar{F}_{12} = -\bar{F}_{21}$So our pair becomes $\bar{F}_{12}(\bar{r}_1 - \bar{r}_2)$
$\bar{r}_1 - \bar{r}_2$ is the vector joining 2 to 1 and is parallel to $\bar{F}_{12}$ as the force is diercted between the particles

So we have $\bar{F}_{12} \times (\bar{r}_1 - \bar{r}_2) = 0$
The pair of terms vanishes
Applying this to all the pairs , we see $\frac{d}{dt}\bar{L}_{tot} = 0$ if no forces act

Applying external forces

$$\frac{d}{dt}\bar{L}_{tot} = \sum_i{}^{'} \bar{r}_i \times \bar{F}_i$$

We have just shown that the effects of any internal forces cancel out, we only need to consider external forces

$$\frac{d}{dt}\bar{L}_{tot} = \sum_i{}^{'} \bar{r}_i \times \bar{F}_i^{ext}$$

Recall the moment of a force aboutsome point is given by $\bar{m} = \bar{r} \times \bar{F}$
So we have

$$\frac{d}{dt}\bar{L}_{tot} = \sum_i{}^{'} \bar{m}^{ext}$$

= net applied moment
Some properties in rotational motion have analogies in linear motion

<u>Transltion table</u>

<u>Linear dynamics</u>	<u>Rotational dynamics</u>
Mass m	Moment of intertia
Velociy	Angular velocity
Force	Moment
N2	
Ke	

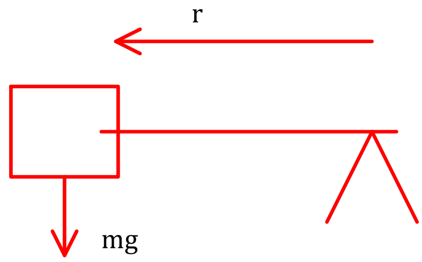
Warning: do not push this too far does not follow for momentum

Recap

Angular momentum of extended objects
In general L is not parallel to ω
For symmetric objects rotating about natural axes, we find
$\bar{L} = I\bar{\omega}$
I= relevant moment of inertia

Tops/gyroscope
Consider a symmetric mass that is free to rotate about an axle. The axle itself is also pivoted, so can change orientation.

The cylinder rotates about its axis
$\bar{L} = I\bar{\omega}$
$\bar{L}$ and $\bar{\omega}$ are both parallel to the axel



Consider an instant when the axle points along the x axis

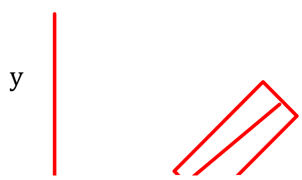
Gravitational force
$$\bar{F} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$
$\bar{r} = \begin{pmatrix} -R \\ 0 \\ 0 \end{pmatrix}$ Where R is the distance of the centre of mass of the gyroscope from the support

The gravitational force generates a moment about the support given by
$$\bar{m} = \bar{r} \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ -R & 0 & 0 \\ 0 & 0 & -mg \end{vmatrix} = \begin{pmatrix} 0 \\ -mgR \\ 0 \end{pmatrix}$$
The moment is perpendicular to $\bar{r}$ and $\bar{F}$ i.e. horizontal and perpendicular to axle
Recall $\frac{d\bar{l}}{dt} = \bar{m} \Rightarrow \frac{d\bar{l}}{dt} = \begin{pmatrix} 0 \\ -mgR \\ 0 \end{pmatrix} \Rightarrow \bar{l}$ changes in its y component => gyroscope moves sideways, not down

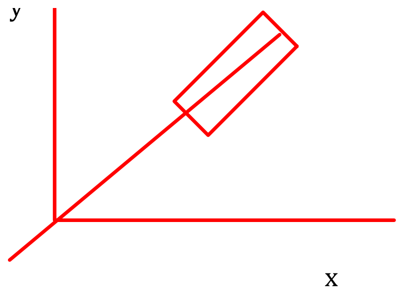
This motion is called precession
Weird-movement is perpendicular to force
Spinning things are more interesting
[bobbing motion is mutation-harder to do sums for]

Pure precession
We will look for a solution where the axle rotates in a horizontal plane at constant angular speed Ω
(N.B. the cylenter rotates at angular speed ω- typically $\omega \gg \Omega$)

We need to show that this motion satisfies
$\frac{d\bar{l}}{dt} = \bar{m}$ at any instant
We need to consider the axle at some arbitrary angle in the x-y plane
From above



If the axel makes an angle θ to the x-axis
$\bar{L} = \begin{pmatrix} L\cos\theta \\ \ldots \end{pmatrix}$



If the axel makes an angle θ to the x-axis

$$\bar{L} = \begin{pmatrix} Lcos\theta \\ Lsin\theta \\ o \end{pmatrix}$$

Similarly

$$\bar{r} = \begin{pmatrix} Rcos\theta \\ Rsin\theta \\ o \end{pmatrix}$$

For pure precession, $\theta = \Omega t + \theta_0$ (for simplicity take $\theta = 0$ at t=0)

The moment acting

$$\bar{m} = \bar{r} \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ Rcos\Omega t & RSin\Omega t & 0 \\ 0 & 0 & -mg \end{vmatrix} = \begin{pmatrix} -mgRsin\Omega t \\ mgRcos\Omega t \\ 0 \end{pmatrix}$$

And

$$\frac{d\bar{l}}{dt} = \begin{pmatrix} -\Omega Lsin\Omega t \\ \Omega Lcos\Omega t \\ 0 \end{pmatrix}$$

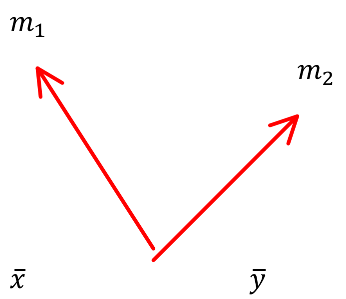
This is a solution iff $\Omega L = mgR \Rightarrow \Omega = \frac{mgR}{L}$

The Two-Body Problem

11 November 2010

09:13

Consider2 bodies under the influence of some force acting between them



The force on 1 due to 2 is $\bar{F}_{12}$

By newton 2: $m_1\ddot{\bar{x}} = \bar{F}_{12}$ (1)

$m_2\ddot{\bar{x}} = \bar{F}_{21}$ *(2)*

6 Coupled second order differential equation

By newton 3 $\bar{F}_{12} = -\bar{F}_{21}$

So (1)+(2) $\Rightarrow m_1\ddot{\bar{x}} + m_2\ddot{\bar{x}} = 0$

Integrating

$\int (m_1\ddot{\bar{x}} + m_2\ddot{\bar{x}})dt = m_1\dot{\bar{x}} + m_2\dot{\bar{x}} = \bar{v}_0$

Integrate again

$\int (m_1\dot{\bar{x}} + m_2\dot{\bar{x}})dt = m_1\bar{x} + m_2\bar{x} = \bar{v}_0 t + \bar{x}_0$

Divide by $m_1 + m_2$

$$\frac{m_1\bar{x} + m_2\bar{x}}{m_1 + m_2} = \frac{\bar{v}_0 t + \bar{x}_0}{m_1 + m_2}$$

$$\underbrace{\frac{m_1\bar{x} + m_2\bar{x}}{m_1 + m_2}}_{Centre\ of\ mass\ of\ the\ system}$$

moves with a constant velocity

C of M moves at constant velocity

Two body problem

26 November 2010
10:15

Steps 1-4 reduce the problem to the relative motion of the two bodies in 2D
$\bar{r} = \bar{x} - \bar{y}$
$\bar{r} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$
$\mu\ddot{\bar{r}} = \bar{F}_{12}(\bar{r}) \Rightarrow \mu \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ 0 \end{pmatrix} = -\frac{Gm_1m_2}{(x_1^2 + x_2^2)^{\frac{3}{2}}} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$
In cartasian coords, this is a mess. Instead we work in polar coordinates as one of these is the distance
To keep things conceptually simple, we will rewrite the components of equation in terms of r and θ
2 equations
$\mu\ddot{x}_1 = -\frac{Gm_1m_2}{(x_1^2 + x_2^2)^{\frac{3}{2}}} x_1$
$\mu\ddot{x}_2 = -\frac{Gm_1m_2}{(x_1^2 + x_2^2)^{\frac{3}{2}}} x_2$
Scalar equations
From the definition of polar coords

$x_1 = r\cos\theta$	$x_2 = r\sin\theta$

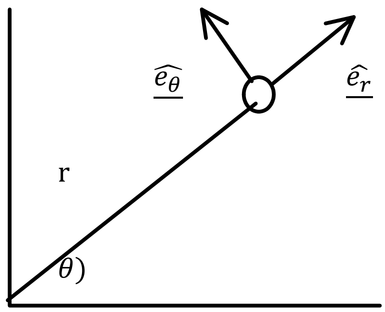
$$\boxed{\begin{gathered} Stroud\ P7{:}\ product\ rule\ f12\ p625 \\ chain\ rule\ f6\ and\ 7\ p623 \end{gathered}}$$

We need $\ddot{x}_1$ and $\ddot{x}_2$
Using $x_1 = r\cos\theta$
$\dot{x}_1 = \frac{d}{dt}(x_1) = \frac{d}{dt}(r\cos\theta) = \left(\frac{d}{dt}r\right)\cos\theta + r\left(\frac{d}{dt}(\cos\theta)\right) = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$
$\frac{d}{dt}\cos\theta(t) = \frac{d\theta}{dt}\frac{d}{d\theta}(\cos\theta)$
We have
$\dot{x}_1 = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$
So
$\ddot{x}_1 = \ddot{r}\cos\theta - \dot{r}\sin\theta\,\dot{\theta} - \dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta}$

$\ddot{x}_1 = \ddot{r}\cos\theta - 2\dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta}$
Similarly
$x_2 = r\sin\theta$
$\dot{x}_2 = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$
$\ddot{x}_2 = \ddot{r}\sin\theta + \dot{r}\cos\theta\,\dot{\theta} + \dot{r}\cos\theta\,\dot{\theta} - r\sin\theta\,\dot{\theta}^2 + r\cos\theta\,\ddot{\theta}$
So
$\ddot{x}_1 = \ddot{r}\cos\theta - 2\dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta}$
$\ddot{x}_2 = \ddot{r}\sin\theta + 2\dot{r}\cos\theta\,\dot{\theta} - r\sin\theta\,\dot{\theta}^2 + r\cos\theta\,\ddot{\theta}$
We can plug these into our 2 equasions
$(a)\quad \mu\left[\ddot{r}\cos\theta - 2\dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta}\right] = -\frac{Gm_1m_2}{r^3}(r\cos\theta)$
$(b)\quad \mu\left[\ddot{r}\sin\theta + 2\dot{r}\cos\theta\,\dot{\theta} - r\sin\theta\,\dot{\theta}^2 + r\cos\theta\,\ddot{\theta}\right] = -\frac{Gm_1m_2}{r^3}(r\sin\theta)$
Now for the "magic"
2 useful things: $\sin^2\theta + \cos^2\theta = 1$

First take $\cos\theta \times (a) + \sin\theta\ (b)$
$\mu[\ddot{r} - r\dot{\theta}^2] = -\frac{Gm_1m_2}{r^2}$
Next take $\sin\theta\ (a) - \cos\theta\ (b)$
$\mu[-2\dot{r}\dot{\theta} - r\ddot{\theta}] = 0$
The magic -> our equations reduce to
$\mu[\ddot{r} - r\dot{\theta}^2] = -\frac{Gm_1m_2}{r^2}$
$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

Was this magic?
No! by picking different set of axes, we can send $\theta - \theta + const$
We don't expect to see θ(undifferentiated) in the equation of motion
Why these two combinations
The combinations we are used to are in fact the radial and tangential components of he equation of motion



These are "natural directions in polar coordinates
Why didn't we work in these directions from the start?
The radial and tangential directions depend on position
here be dragons
Our rule for differentiating a vector by differentiating its components only holds if the basis vectors are constants -> this is true for Cartesian coords (I, j and k are constant) but not for polar

No we need to solved

$\mu[\ddot{r} - r\dot{\theta}^2] = -\frac{Gm_1m_2}{r^2}$
A
$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$
B
Start with
$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$
$\Rightarrow (2r\dot{r})\dot{\theta} + r^2\ddot{\theta} = 0$
$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$
Using product rule
$\Rightarrow r^2\dot{\theta} = const$
Can the constant l $\Rightarrow$ $\boxed{r^2\dot{\theta} = l}$ C

This is conservation of angular momentum
Now consider A
$\mu[\ddot{r} - r\dot{\theta}^2] = -\frac{Gm_1m_2}{r^2}$

Use C to eliminate $\dot{\theta}$
Type equation here.

$$\mu\left[\ddot{r}-\frac{l^2}{r^3}\right]=-\frac{Gm_1m_2}{r^2}$$

Multiply by $\dot{r}$

$$\mu\left[\dot{r}\ddot{r}-\frac{l^2\dot{r}}{r^3}\right]=-\frac{Gm_1m_2\dot{r}}{r^2}$$

$$\Rightarrow \frac{d}{dt}\left[\mu\frac{\dot{r}^2}{2}+\mu\frac{l^2}{2r^2}\right]-Gm_1m_2\frac{d}{dt}\left[-\frac{1}{r}\right]$$

$$\Rightarrow \underbrace{\frac{1}{2}\mu\dot{r}^2+\frac{1}{2}\mu\frac{l^2}{r^2}}_{\substack{\frac{1}{2}\mu\dot{r}^2+\frac{1}{2}\mu\left(r\dot{\theta}\right)^2 \\ Using\ C}} \quad \underbrace{-\frac{Gm_1m_2}{r}}_{PE}=const$$

$\dot{r}\ and\ r\dot{\theta}$ are radial and tangential components of the velocity

$$\Rightarrow speed^2=\dot{r}^2+\left(r\dot{\theta}\right)^2$$

so

$$\frac{1}{2}\mu\left(\dot{r}^2+\left(r\dot{\theta}\right)^2\right)=\frac{1}{2}\mu(speed)^2=Ke$$

So the equation

$$\frac{1}{2}\mu\left(\dot{r}^2+\frac{l^2}{r^2}\right)-\frac{Gm_1m_2}{r}=const$$

We can interpret as energy conservation

<u>Cunning trick</u>
We will work with

$$\frac{1}{2}\mu\dot{r}^2+\frac{1}{2}\mu\frac{l^2}{r^2}-\frac{Gm_1m_2}{r}=C\mu$$

Let $u=\frac{1}{r}$ and look for an equation in $\frac{d}{d\theta}$ rather than $\frac{d}{dt}$

Given

$$r=\frac{1}{u}\Rightarrow\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}\quad\left\{\begin{array}{l} thinking\ of\ u\ as \\ a\ function\ of\ \theta\ and \\ using\ the\ chain\ rule\end{array}\right.$$

$$=-r^2\dot{\theta}\frac{du}{d\theta}=-l\frac{du}{d\theta}$$

Using C

D then becomes

$$\frac{1}{2}\mu l^2\left(\frac{du}{d\theta}\right)^2+\frac{1}{2}\mu l^2u^2-Gm_1m_2u=C\mu\Rightarrow\frac{1}{2}\left(\frac{du}{d\theta}\right)^2=\frac{Gm_1m_2u}{\mu r^2}-\frac{1}{2}u^2+\frac{C}{r^2}=\alpha u-\frac{1}{2}u^2+\frac{c}{r^2}$$

$$\Rightarrow\frac{du}{d\theta}=\sqrt{2\alpha u-u^2+\frac{2C}{l^2}}$$

$$\Rightarrow\frac{du}{\sqrt{\frac{2C}{l^2}+\alpha^2-(U-\alpha)^2}}=d\theta$$

$$\Rightarrow sin^{-1}\left(\frac{U-\alpha}{\sqrt{\frac{2C}{l^2}+\alpha^2}}\right)=\theta-\theta_0$$

$$\frac{du}{\sqrt{\frac{2C}{l^2}+\alpha^2}\sqrt{1-\frac{(U-\alpha)^2}{\frac{2C}{l^2}+\alpha^2}}}=d\theta$$

Make substitution

$$\sin\phi = \frac{U-\alpha}{\sqrt{\frac{2C}{l^2}+\alpha^2}}$$

$$\cos\phi\, d\phi = \frac{du}{\sqrt{\frac{2C}{l^2}+\alpha^2}}$$

$$d\theta = \frac{\cos\phi\, d\phi}{\cos\phi}$$

$$\int d\theta = \int dp$$

$$\Rightarrow \theta - \theta_0 = \phi$$

Then

$$\sin\phi = \frac{U-\alpha}{\sqrt{\frac{2C}{l^2}+\alpha^2}} = \sin(\theta-\theta_0)$$

$$\Rightarrow U - \alpha = \sqrt{\frac{2C}{l^2}+\alpha^2}\sin(\theta-\theta_0)$$

$$U = \frac{1}{r} = \alpha + \sqrt{\frac{2C}{l^2}+\alpha^2}\sin(\theta-\theta_0)$$

Recap: the orbit equation

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$$\frac{1}{r} = \alpha + \sqrt{\frac{2c}{l^2} + \alpha^2} \sin(\theta . \theta_0)$$

$$\alpha = \frac{Gm_1 m_2}{\mu l^2} = \frac{G}{l^2}(m_1 + m_2)$$

As

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

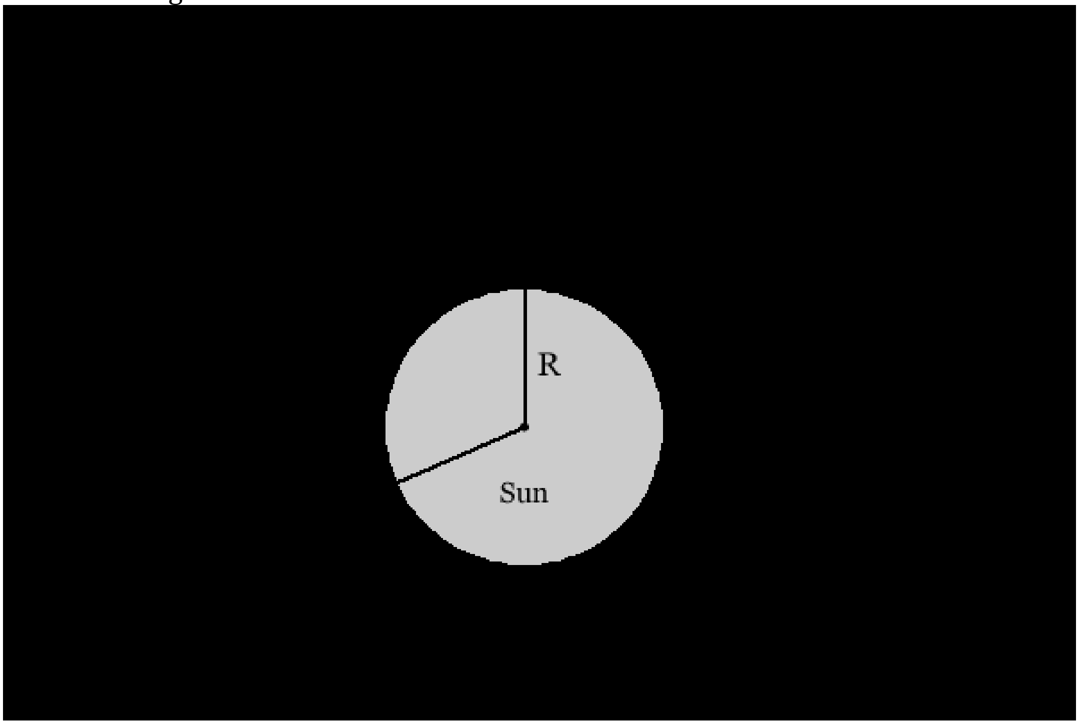
$$\left.\begin{aligned} l &= r^2 \dot{\theta} \;\; angular\ momentum\ per\ unit\ \mu \\ c &= \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 - \frac{Gm_1 m_2}{\mu r} \;\; energy\ per\ unit\ \mu \end{aligned}\right\} constants$$

Obtaining l and c

1. <u>Closed orbits</u>
   If c<0 we have gravitationally bound system and he orbit is closed. Consider a satellite in a closed orbit around a planet. Let the closest approach distance be r_{min} and maximum distance r_{max}.
   The mass of the planet $m_p \gg m_{sat}$ and we take the centre of mass to be the centre of the planet
   At r_{min} and r_{max}, the velocity is perpendicular to radial direction at r_{min}: $l = r^2\dot{\theta} = r(r\dot{\theta})$.
   $(r\dot{\theta})$ is tangential component of velocity
   $$\left.\begin{aligned} at\ r_{min} \quad l &\rightarrow r_{min} v_{max} \\ similarly\ at\ r_{max} \quad l &= r_{max} v_{min} \end{aligned}\right\} conservation\ of\ angular\ momentum$$
   L is constant $\Rightarrow r_{min} v_{max} = r_{max} v_{min} \Rightarrow v_{max} = \left(\frac{r_{max}}{r_{min}}\right) v_{max} \qquad (A)$
   Now consider
   $$c = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 - \frac{Gm_p}{\mu r}$$
   m_p as mass of planet $\gg m_{sat}$
   At r_{min}, $c = \frac{1}{2} v_{max}^2 - \frac{Gm_p}{r_{\min}}$
   At r_{max}, $c = \frac{1}{2} v_{\min}^2 - \frac{Gm_p}{r_{max}}$
   As c is constant ^
   $$\frac{1}{2} v_{max}^2 - \frac{Gm_p}{r_{\min}} = \frac{1}{2} v_{\min}^2 - \frac{Gm_p}{r_{max}} \qquad (B)$$
   Given r_{min}, r_{max} *and* m_p (A) and (B) are a pair of simultanious equations in v_{min} *and* v_{max} solve for v_{min} *and* v_{max} ………..
2. Open orbits
   If c>0 at two values of $\theta, r \rightarrow \infty$ and the orbit is open
   Classic application is the gravitational sling shot
   Consider a satellite performing a sling shot
   $m_p \gg m_{sat} \rightarrow$ the centre of mass is just the centre of the planet. At this stage, treat the planet as stationary.

   Let the satellite have velocity <u>v</u> at large distances from the planet. The <u>Impact parameter</u> is what the closest approach distance would be if there was no gravitational interaction
   We need to find $l = r^2\dot{\theta} = rv_{tangential} = rv\sin\theta$
   From the diagram,
   $sin\,\theta = \frac{b}{r} \Rightarrow b = r\sin\theta \Rightarrow l = vb$
   We also need
   $$c = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 - \frac{Gm_p}{\mu r}$$

At larger PE term->0 and $c \to \frac{1}{2}v^2(at\ larger)$
Deflection angle



$\theta_{in} and\ \theta_{out}$ correspond to $r \to \infty \Rightarrow \alpha + \sqrt{\frac{2c}{l^2} + \alpha^2}\sin(\theta\theta_0) = 0$

$$\Rightarrow \sin(\theta\theta_0) = -\frac{1}{\sqrt{\frac{2c}{\alpha^2 l^2} + 1}}$$

For c~0 require
$\sin\theta_{in}, \sin\theta_{out} \sim -1 \Rightarrow \theta_{in}, \theta_{out} \propto \frac{2\pi}{2} \Rightarrow$
$satellite\ componant\ in\ \ nearly\ opposite\ direction$

Creating the parameters

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2. Open orbits
<u>eg</u>
Cassini & venus
A mission planner suggests the following slingshot:
satellite cassini $m_{Cassini} = 5.6 \times 10^3 kg$
Planet venus $m_{Venus} = 4.9 \times 10^{24} kg$
Speed at large distance $v = 5 \times 10^3 ms^{-1}$
Impact parameter $b = 10^7\ m$
Find deflection angle
Why was the planner sacked?

For deflection angle we need to find

$$\sin\theta = -\frac{1}{\sqrt{1+\frac{2c}{l^2\alpha^2}}}$$

Use

$$c = \frac{1}{2}v^2, l = vb, \alpha = \frac{Gm_1m_2}{\mu l^2} = \frac{G(m_1+m_2)}{l^2}$$

Subing in

$$\frac{2c}{l^2\alpha^2} = \frac{v^2}{l^2\left(\frac{G^2(m_{Venus})^2}{l^{2^2}}\right)} = \frac{v^2l^2}{G^2m_{Venus}^2} = \frac{v^4b^2}{G^2m_{Venus}^2} = 0.50$$

$$\Rightarrow \sin\theta = -\frac{1}{\sqrt{1.59}}$$

Calculate $\theta = -52.5°$ this means our solution is $360 - 52.5 = 307.5°$ other solution is $180 + 52.5 = 232.5$

<u>Closest approach & maximum speed</u>
We know

$$\frac{1}{r} = \alpha\left|1+\sqrt{1+\frac{2c}{l^2\alpha^2}}\sin\theta\,\theta_0\right|$$

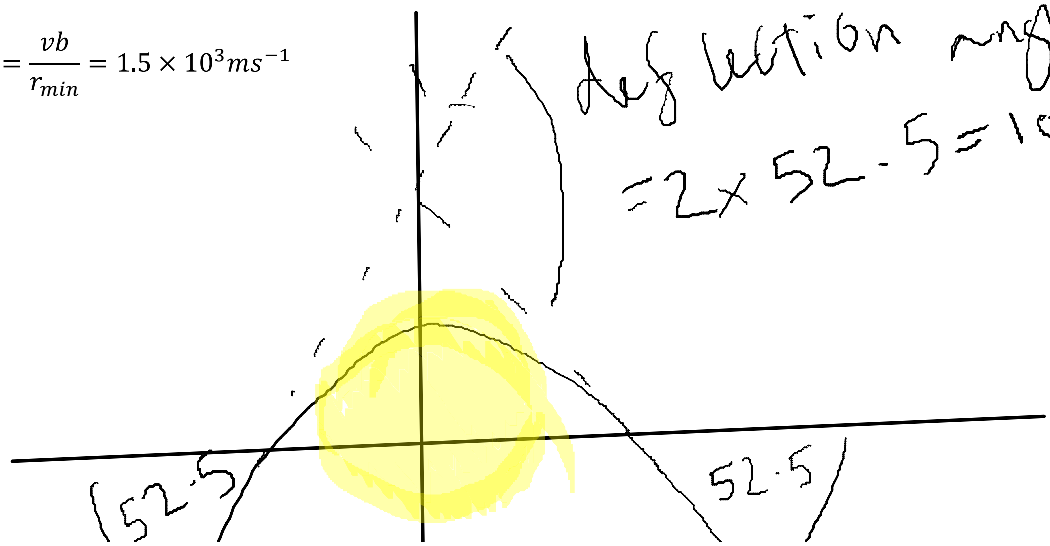
At the closest approach, r is a minimum,$\Rightarrow \frac{1}{r}$ is at its maximum value $\Rightarrow \sin\theta\theta_0$

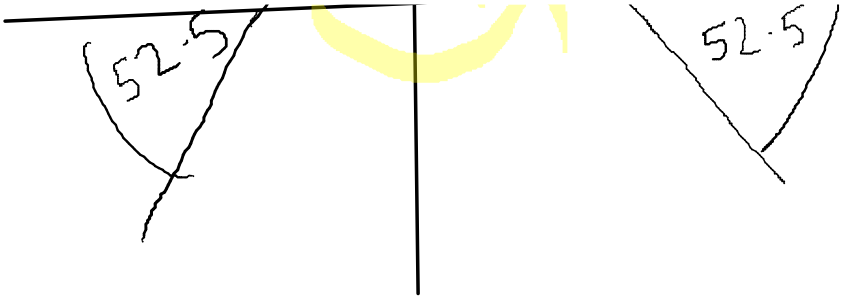
$$\Rightarrow r_{min} = \frac{1}{\alpha\left|1+\sqrt{1+\frac{2c}{l^2\alpha^2}}\right|} = 3.4 \times 10^6 m$$

Radius of venus 6×10^6
Also, at closest approach the velocity is in the tangential direction so (just as in the case of closest orbits)

$$l = r_{min}v_{max} \Rightarrow v_{max} = \frac{vb}{r_{min}} = 1.5 \times 10^3 ms^{-1}$$

deflection angle $= 2 \times 52.5 = 105°$




52.5
52.5

Kepler's laws

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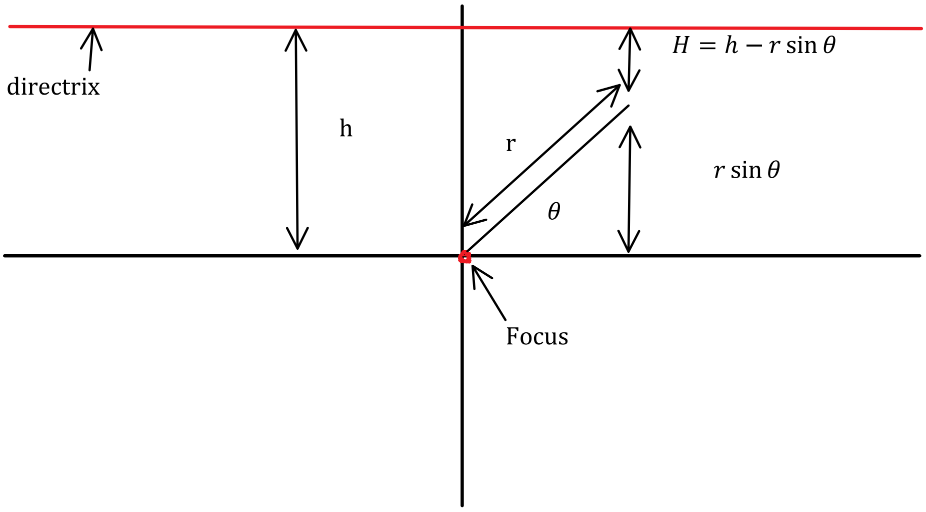
Ellipse

Consider the locus of a point moving such that its distance from a fixed point is a trivial multiple of its distance from a fixed straight line
The fixed point is the focus
The fixed line is the directrix
Use coordinates with origin at the focus



We are interested in points satisfying r=eH
Where e is constant (eccentricity

$\Rightarrow r = e(h - r\sin\theta) \Rightarrow \frac{r}{2} = h - r\ \sin\theta \Rightarrow \frac{1}{e} = \frac{h}{r} - \sin\theta \Rightarrow \frac{1}{e} + \sin\theta = \frac{1}{r} \Rightarrow \boxed{\frac{1}{r} = \frac{1}{he}(1 + e\sin\theta)}$

Compare with the orbit equation $(set\ \theta_0 = 0)$

$\frac{1}{r} = \alpha[1 + \sqrt{1 + \frac{2c}{l^2\alpha^2}}\sin\theta$

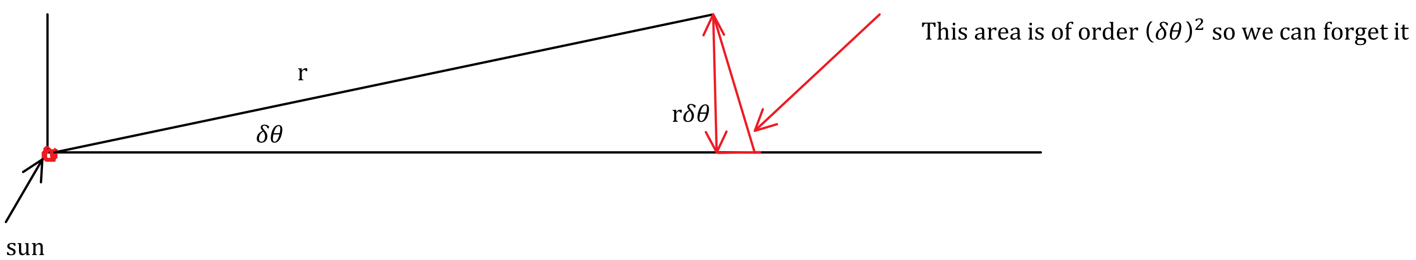
We have the same form

$\alpha \equiv \frac{1}{he} \qquad e \equiv \sqrt{1 + \frac{2c}{l^2\alpha^2}}$

For c<0 we have closed orbits, e<1 if e<1 the locus describes an ellipse
The focus of the ellipse is athe origin
For our two body system, the centre of mass is at the origin
=>our bodies move in elliptical orbits about the centre of ass
For the sun/planet system the centre of mass is essentially at the centre of the sun => we recover kepler 1 for planetary motion
-> motion on an elipse with the sun at the focus (kepler's law of orbits)

Law of areas

Line conecting planet to sun sweeps out equal areas in equal times



Area swept out=area of triangle on left

$$=\frac{1}{2}r \times r\delta\theta = \frac{1}{2}r^2\frac{\delta\theta}{\delta t}\delta t \to \frac{1}{2}r^2\dot{\theta}\delta t \; (for\ small\ \delta t, \delta\theta)$$

$$=\frac{1}{2}\delta t = (constant)\delta t$$

⇒area law follows from angular momentum conservation (i.e. that l=constant*)*