Quantum Mechanics

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13:54

Evaluation
2 mid term exams - 20% of final grade
One final - 80% of final grade

Schrodinger eq

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Qm- 1900
Schrodinger- 1925

Eq for a function- Wavefunction $\psi(x, y, z, t)$
Eq in partial derivatives [differential of]

For a particle of mass m under the influence of a potential V_x

$$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\psi(x,t) + V_x\psi(x,t) = i\hbar\frac{\delta}{\delta x}\psi(x,t)$$

Suppose that

$$\psi(x,t) = e^{-\frac{iE}{\hbar}t}\psi(x)$$

(stationary state)

And put it in schrodinger eq

$$\frac{\delta}{\delta t}\psi(x,t) = -\frac{i}{\hbar}Ee^{-\frac{i}{\hbar}Et}\psi(x)$$

$$i\hbar\frac{\delta}{\delta t}\ \psi(x,t) = E\ e^{-\frac{iE}{\hbar}t}\psi(x)$$

$$\frac{\delta^2}{\delta x^2}\psi(x,t) = \frac{\hbar}{m}e^{-\frac{iE}{\hbar}t}\ \psi''(x)$$

$$V\psi = e^{-\frac{iE}{\hbar}t}v\psi(x)$$

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\psi'' + (V-E)\psi = 0$$

$$\psi(x,t) = Ae^{-\frac{iE}{\hbar}t}e^{-\lambda x^2}$$

1) $i\hbar\frac{\delta}{\delta t}\ \psi(x,t)$
2) $\frac{\delta\psi}{\delta x}$
3) $-\frac{\hbar^2}{2m}\frac{\delta^2\psi}{\delta x^2}$

1) $AE\ Ae^{-\frac{iE}{\hbar}t}e^{-\lambda x^2}$
2) $Ae^{-\frac{iE}{\hbar}t}(-2\lambda x)e^{-\lambda x^2}$
3) $\frac{\hbar^2}{2m}Ae^{-\frac{iE}{\hbar}t}2\lambda(2\lambda x^2e^{-\lambda x^2} - e^{-\lambda x^2})$

$$-\frac{\hbar}{2m}Ae^{-\frac{iE}{\hbar}t}e^{-\lambda x^2}(4\lambda^2x^2 - 2\lambda) + V\ A\ e^{-\frac{iE}{\hbar}t}e^{-\lambda x^2} = AEe^{\frac{iE}{\hbar}t}e^{-\lambda x^2}$$

$$-\frac{\hbar^2}{2m}(4\lambda^2x^2 - 2\lambda) + V_x - E = 0$$

$$V = \alpha_1x^2 + \alpha_2$$

$\psi(x,t)$ is the wavefuncction of a system
$\psi^*(x,t) \times \psi(x,t) = |\psi(x)|^2$ →probability of the system to be between (x,x+dx)

What we need to impose on a wavefunction?

1. Solves schrodinger eq
   $$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\psi(x,t) + V_x\psi(x,t) = i\hbar\frac{\delta}{\delta x}\psi(x,t)$$
2. It has to be differentiable
3. Simple valued
4. Normalizable $\int_{-\infty}^{\infty} dx|\psi(x)|^2 = 1$

$\psi(x,t) = Ae^{-\frac{iE}{\hbar}t}e^{-\lambda x^2}$
For a particular V(x)-> solve schrodinger eq
$\psi\psi^* = |A|^2 e^{-2\lambda x^2}$

$$|A|^2 \int_{-\infty}^{\infty} e^{-2\lambda x^2} dx = |A|^2\sqrt{\frac{\pi}{2\lambda}}$$

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}$$
$$\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{2\lambda^3}}$$

$\psi(x,t) = Ae^{-\frac{iE}{\hbar}t}e^{-\lambda x^2}$
For a particular V(x)-> solve schrodinger eq
$\psi\psi^* = |A|^2 e^{-2\lambda x^2}$

$$|A|^2 \int_{-\infty}^{\infty} e^{-2\lambda x^2} dx = |A|^2\sqrt{\frac{\pi}{2\lambda}}$$

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$\Psi(x,t) = e^{-\frac{i}{\hbar}Et}\psi(x)$

Stationary states

$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$

$-\frac{\hbar^2}{2m}\psi'' + (V-E)\psi = 0$

Useful today

$\int_{-\infty}^{\infty} e^{-\sigma x^2} dx = \sqrt{\frac{\pi}{\sigma}}$

$\int_{-\infty}^{\infty} x^2 e^{-\sigma x^2} dx = \sqrt{\frac{\pi}{4\sigma^3}}$

$\int_{-\infty}^{\infty} x e^{-x^2\sigma} dx = 0$

1. Continuous, differentiable
2. Normalizable

$$\int_{-\infty}^{\infty} dx\, \psi(x,t)\psi^*(x,t) = \int_{-\infty}^{\infty} dx\, |\psi|^2 = 1$$

$\psi(x,t) = A\, e^{-\frac{i}{\hbar}Et} e^{-\lambda x^2}$

Plug this in to the schrodinger eq to find a solution $\rightarrow V \sim (x^2 + p)$

λ positive real number

A Some number

$|\psi|^2 = |A|^2 e^{-2\lambda x^2}$

Calculate

$< x > \;= \int_{-\infty}^{\infty} \psi^*(x,t) x\, \psi(x,t) dx = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-2\lambda x^2} x\, dx = 0$

$< x^2 > \;= \int_{-\infty}^{\infty} \psi^* x^2 \psi = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-2\lambda x^2} x^2 dx = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}} \sqrt{\frac{\pi}{4*8\lambda^3}} = \sqrt{\frac{1}{16\lambda^2}}$

$(\Delta x) = \sqrt{< x^2 > - < x >^2} = \sqrt{\frac{1}{4\lambda}}$

$f(x) = e^{-\sigma x^2}$

$f' = -2\sigma x\, e^{-\sigma x^2}$

$f'' = (-2\sigma + 4\sigma^2 x^2) e^{-\sigma x^2}$

<u>Useful today</u>

$\int_{-\infty}^{\infty} e^{-\sigma x^2} = \sqrt{\frac{\pi}{\sigma}}$

$\int_{-\infty}^{\infty} x^2 e^{-\sigma x^2} dx = \sqrt{\frac{\pi}{4\sigma^3}}$

$\int_{-\infty}^{\infty} x^{2m+1} e^{-\sigma x^2} dx = 0$

$\psi(x,t) = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}} e^{-\frac{i}{\hbar}Et} e^{-\lambda x^2}$

1. $|\psi|^2 = |A|^2 e^{-2\lambda x^2} \rightarrow imposed \rightarrow \int_{-\infty}^{\infty} |\psi|^2 dx = 1 \rightarrow A = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}$
2. $< x > \;= 0 = \int_{-\infty}^{\infty} \psi^* x \psi dx = |A|^2 \int_{-\infty}^{\infty} x e^{-2\lambda x^2} dx = 0$
3. $< x^2 > \;= \int_{-\infty}^{\infty} \psi^* x^2 \psi dx \;= |A|^2 \int e^{-2\lambda x^2} x^2 dx = \frac{1}{4\lambda}$

$$\Delta x = \sqrt{< x^2 > - < x >^2} = \frac{1}{2\sqrt{\lambda}}$$

$\lambda \rightarrow \infty, \Delta x \rightarrow 0$

$\lambda \rightarrow 0, \Delta x \rightarrow \infty$

<u>Define</u>

Momentum operator

$\vec{p} = \frac{\hbar}{i}\frac{\delta}{\delta x}$

$$\nabla f = \frac{\delta f}{\delta x}\imath + \frac{\delta f}{\delta y}\jmath$$

$\vec{p}\psi(x,t) = \frac{\hbar}{i}\frac{\delta}{\delta x}\psi(x,t)$

$p^2\psi(x,t) = pp\psi$

$\frac{\hbar}{i}\frac{\delta}{\delta x}\left(\frac{\hbar}{i}\frac{\delta}{\delta x}\psi\right) = -\hbar^2\frac{\delta^2}{\delta x^2}\psi$

$< \vec{p} > \ = \int_{-\infty}^{\infty}\psi^*\left(\frac{\hbar}{i}\right)\frac{\delta}{\delta x}\psi dx = \int_{-\infty}^{\infty}\frac{\hbar}{i}\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}}(-2\lambda x)e^{-2\lambda x^2}dx = \frac{\hbar}{i}\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}(-2\lambda x)e^{-2\lambda x^2}dx$

$= \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}}\frac{\hbar}{i}(-2\lambda)\int_{-\infty}^{\infty}xe^{-2\lambda x^2}dx = 0$

$< p^2 > \ = \int_{-\infty}^{\infty}\psi^*(-\hbar^2)\frac{\delta^2}{\delta x^2}\psi dx = \int_{-\infty}^{\infty}\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}e^{\frac{1}{\hbar}Et}e^{-\lambda x^2}(-\hbar^2)\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}e^{-\frac{1}{\hbar}Et}(-2\lambda + 4\lambda^2x^2)e^{-\lambda x^2}dx$

$= \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}(-\hbar^2)\int_{-\infty}^{\infty}(-2\lambda + 4\lambda^2x^2)e^{-2\lambda x^2}dx = -\hbar^2\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}(-2\lambda + 4\lambda^2x^2)e^{-2\lambda x^2}dx$

$= -\hbar^2\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{2}}[-2\lambda\sqrt{\frac{\pi}{2\lambda}} + 4\lambda\sqrt{\frac{\pi}{4*8\lambda^2}}$

$\psi(x,t) = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}e^{-\frac{i}{\hbar}Et}e^{-\lambda x^2}$

$\Delta p = \sqrt{< p^2 > - < p >^2} = \hbar\sqrt{\lambda}$

$\Delta x = \frac{1}{2\sqrt{x}}$

$\Delta p = \hbar\sqrt{\lambda}$

$\lambda \to \infty\ \Delta p \to \infty$

$\lambda \to 0\ \ \Delta p \to 0$

$\Delta x\Delta p \sim \frac{\hbar}{2}$

Uncertainty principle

$x = x_0 + v_0t + \frac{at^2}{2}$

$f = A\cos\omega x + B\sin\omega x$

$f'' = -A\omega^2\cos\omega x - B\omega^2\sin\omega x$

$f'' + \omega^2 f = 0$

Oscillator

$f(x) = A\cos\omega x + B\sin\omega x$

$f(x=0) = 0 \to A*1 + B*0 = 0 \Rightarrow \boxed{A=0}$

$f(x=L) = 0 \to B\sin\omega L = 0\left\{\begin{matrix} B=0 \\ \sin\omega L = 0\end{matrix}\right.$

$\sin\omega L = 0$

$\omega L = k\pi$

$\omega = \frac{k\pi}{L}$

$f = B_1\sin\left(\frac{k\pi}{L}x\right) + B_2\sin\left(\frac{k\pi}{L}x\right)$

$-\frac{\hbar^2}{2m}\frac{\delta^2\psi}{\delta x^2} + V\psi = i\hbar\frac{\delta}{\delta t}\psi$

$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\psi = i\hbar\frac{\delta}{\delta t}\psi$

$\psi(x,t) = \psi(x)e^{-\frac{i}{\hbar}Et}$

$-\frac{\hbar^2}{2m}e^{-\frac{i}{\hbar}Et}\psi''(x) = i\hbar\left(-\frac{i}{\hbar}\right)Ee^{-\frac{i}{\hbar}Et}\psi(x)$

$-\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$

$\psi''(x) + \omega^2\psi(x) = 0$

$\omega^2 = \frac{2mE}{\hbar^2}$

$\psi(x) = A\cos\omega x + B\sin\omega x$
$\psi(x=0) = 0$
$\psi(x=L) = 0$

$\psi(x) = B\sin\omega x$

$$\omega = \sqrt{\frac{2mE}{\hbar^2}} = \frac{\pi k}{L}$$

$$\frac{2mE}{\hbar^2} = \frac{\pi^2 k^2}{L^2}$$

$$E_k = \frac{\pi^2 k^2 \hbar^2}{2mL^2}$$

I | II | III
$\psi = 0$ | | $\psi = 0$
| _ _ _ _ |
L

$$\psi(x,t) = \begin{cases} 0 \; in\ between\ I\ and\ III \\ B_k e^{-\frac{i}{\hbar}E_k t}\sin\left(\frac{k\pi}{L}x\right) \end{cases}$$

$$\boxed{E_k = \frac{\pi^2 k^2 \hbar^2}{2mL^2}}$$

$$\psi(x,t) = \sum_{k=1}^{\infty} B_k e^{-\frac{i}{\hbar}E_k t}\sin\left(\frac{k\pi}{L}x\right)$$

Guitar

$B_k\sin\left(\frac{k\pi}{L}x\right)\ k \in \mathbb{Z}$

$\sum_{k=1}^{\infty} B_k\sin\left(\frac{k\pi}{L}x\right), |B_k| = amplitude\ of\ each\ mode$

$|B_k|^2 = energy\ of\ each\ mode$

$$\psi(x,t) = B_k e^{-\frac{i}{\hbar}E_k t}\sin\left(\frac{k\pi}{L}x\right)$$

$$\psi(x,t) = \sum_{k=1}^{\infty} B_k e^{-\frac{i}{\hbar}E_k t}\sin\left(\frac{k\pi}{L}x\right)$$

Will have a very deep meaning

Condition to impose on the wavefunction

$$\int_{-\infty}^{\infty}\psi\psi^* dx = 1 \rightarrow probability$$

$$E_k = \frac{\pi^2 k^2 \hbar^2}{2mL^2}$$

$$\int_0^L \psi\psi^* = |B|^2 \int_0^L \sin^2\left(\frac{k\pi}{L}x\right) = 1 \Rightarrow B_k = \sqrt{\frac{2}{L}}$$
$$\uparrow\left(\frac{L}{2}\right)$$

Believe me that

$$\int_0^L \psi(x,t)\psi^*(x,t)dx = 1 \rightarrow \sum_{k=1}^{\infty}|c_k|^2 = 1$$

c_k= probability of the "state" of the particle to be that of the k oscillation

Example

Write the wave function for a particle in the infinite square well such that
the probability for its energy to be $E_{10}\ is\ \frac{1}{3}$
the probability for its energy to be $E_{28}\ is \frac{2}{3}$

$$\psi(x,t) = c_{10}\sqrt{\frac{2}{L}}e^{-\frac{2}{\hbar}E_{10}x}\sin\left(\frac{10}{L}\pi x\right) + c_{28}\sqrt{\frac{2}{L}}e^{-\frac{2}{\hbar}E_{28}x}\sin\left(\frac{28}{L}\pi x\right)$$

$$c_{10}^2 = \frac{1}{3}$$

$$c_{28}^2 = \frac{2}{3}6$$

$$E_{28} = \frac{(28)^2\hbar^2\pi^2}{2mL^2}$$

1. Quantum mechanics is not deterministic
2. If you measure (in this case, the "energy") something quite dramatic happens after the measurement

$$\psi(x,t) = \sqrt{\frac{2}{L}}\exp\left(-\frac{i}{\hbar}E_{10}t\right)\sin\left(\frac{10\pi}{L}x\right)$$

Measurement destroys wavefunction

1. Start with harmonic oscillation
2. Complete some of the things that we discussed last time

$$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\psi + V\psi = i\hbar\frac{\delta}{\delta t}\psi$$

Last time we solved it for

Today we will study

$$\boxed{V = \frac{\kappa x^2}{2}}$$

As in previous lectures

<u>Propose</u>

$$\psi(x,t) = e^{-\frac{i}{\hbar}Et}\psi(x)$$

→ we plug this into the schrodinger eq

Check at home

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{\kappa}{2}x^2\psi = E\psi$$

$$\psi'' + x^2\psi = E\psi$$

$$(a^2 - b^2) = (a+b)(a-b)$$

The idea is to factorize the 2nd order eq

Define two <u>operators</u>

$$a_+ = \frac{1}{\sqrt{2m}}\left\{\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right\}$$

$$a_- = \frac{1}{\sqrt{2m}}\left\{\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right\}$$

$$a_+a_- = a_+\frac{1}{\sqrt{2m}}\left\{\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right\}$$

$$\frac{1}{\sqrt{2m}}\left\{\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right\}\frac{1}{\sqrt{2m}}\left\{\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right\}$$

$$= \frac{1}{2m}\left\{-\hbar^2\psi'' - \frac{\hbar}{i}im\omega\frac{d}{dx}(x\psi) + 2m\omega x\frac{\hbar}{i}\psi'' + m^2\omega^2x^2\psi\right\}$$

$$= \frac{1}{2m}\{-\hbar^2\psi'' - \hbar m\omega(\psi + x\psi') + m\omega\hbar x\psi + m^2\omega^2x^2\psi\}$$

$$a_+a_-\psi = -\frac{\hbar^2}{2m}\psi'' - \frac{\hbar\omega}{2}\psi + \frac{m\omega^2}{2}x^2\psi$$

$$a_+a_-\psi = E\psi - \frac{\hbar\omega}{2}\psi$$

$$\omega^2 = \frac{k}{m}$$

$$-\frac{\hbar^2}{2m}\psi'' + \frac{m\omega^2}{2}x^2\psi = E\psi$$

$$a_-a_+\psi = \frac{1}{\sqrt{2m}}\left\{\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right\}\frac{1}{\sqrt{2m}}\left\{\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right\}$$

$$= \frac{1}{2m}\{-\hbar^2\psi'' + \hbar m\omega(x\psi + \psi) - \hbar m\omega x\psi + m^2\omega^2x^2\psi\}$$

$$= -\frac{\hbar^2}{2m}\psi'' + \frac{m\omega^2}{2}x^2\psi + \frac{\hbar\omega}{2}\psi$$

$$\psi(x,t) = \sum_k c_k\sqrt{\frac{2}{L}}e^{-\frac{i}{\hbar}E_kt}\sin\left(\frac{k\pi}{L}x\right)$$

1. E quantized

$$E_k = \frac{\hbar^2\pi^2k^2}{2mL^2}$$

$(c_k)^2$ = probability for the particle to be in k-state

\newcommand{\~new~}{\~old~}

Summary

We study the oscillator in quantum mechanics

This means the schrodinger equation for a particlue under a potential

$$v = \frac{kx^2}{2}$$

$\left(v = \frac{m\omega^2}{2} x^2 \right)$

We proposed a wavefunction

$\psi(x,t) = e^{\frac{i}{\hbar}Et}\psi(x)$

We get

$$\boxed{-\frac{\hbar^2}{2m}\psi''(x) + \frac{m\omega^2}{2}x^2\psi(x) = E\psi(x)} (*)$$

Special function

Very difficult to solve

The problem here is to find

E=possible energy

$\psi(x)$= wavefunction

Actually, there is a nice way olf solving this without solving the eq (*)

To do this we started to study an idea due to heisenberg (st

Last week we introduced two operators

a_+

a_-

$|a_+, a_-| = \hbar\omega$

$|a_-, a_+| = \hbar\omega$

Kinetic + potential

$H = \frac{1}{2}\, mv^2 + V(x)$

$\frac{1}{m}mv^2 = \frac{p^2}{2m}$

$p = mv$

$p^2 = -\hbar^2 \frac{d^2}{dx^2}$

$$\boxed{H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)}$$

At home, show that

$$\boxed{\begin{aligned} H &= a_+a_- + \frac{\hbar\omega}{2} \\ or & \\ H &= a_-a_+ - \frac{\hbar\omega}{2} \end{aligned}}$$

We will know the result for

$|a_+, H| = \text{ show at home } -\hbar\omega a_+$

$|a_-, H| = \hbar\omega a_-$

As a sample let me calculate

$|a_-.H| = a_-H - Ha_- = a_-\left(a_-a_+ - \frac{\hbar\omega}{2}\right) - \left(a_-a_+ - \frac{\hbar\omega}{2}\right)a_-$

$= a_-a_-a_+ - \frac{\hbar\omega}{2}a_- - a_-a_+a_- + \frac{\hbar\omega}{2}a_-$

$= a_-a_-a_+ - a_-a_+a_- =$

The idea is instead of $\boxed{a_-a_+ = -\hbar\omega + a_+a_-}$

$a_-(-\hbar\omega + a_+a_-) - a_-a_+a_- = -\hbar\omega_- \mp a_-a_+a_- - a_-a_+a_- = -\hbar\omega a_-$

Theorem

If you have a wavefunction $\psi_*(x)$

That solves the Schrödinger equation with energy E_*

$H\psi_* = E_*\psi_*$

Then you can construct two other functions

$a_+\psi_*$

$a_-\psi_*$

They will solve the Schrodinger q $\left(\begin{matrix} E_* + \hbar\omega \\ E_* - \hbar\omega \end{matrix} \right|$

$a_+ \rightarrow \textit{satisfies the eq with } E = E_* + \hbar\omega$

$a_- \rightarrow \textit{satisfies the eq with } E = E_* - \hbar\omega$

$H(a_+\psi) = |a_+, H| = (a_+H + \hbar\omega a_+)\psi$

$= a_+E\psi + \hbar\omega a_+\psi = (E + \hbar\omega)(a_+\psi)$

$(a_+\psi) = \psi_+$

remember

$$\boxed{\int_{-\infty}^{\infty} e^{-\sigma x^2} dx = \sqrt{\frac{\pi}{\sigma}}}$$

$f'(x) = \alpha x\, f(x) \rightarrow \frac{1}{f(x)}\frac{df}{dx} = \alpha x$

$\Rightarrow \frac{1}{f(x)}df = \alpha x\, dx$

$\Rightarrow \int \frac{df}{f} = \int \alpha x\, dx$

$$\boxed{\log f = \frac{\alpha x^2}{2} + c}$$

$f'(x) = \alpha x\, f(x) \rightarrow f(x) = Ae^{\frac{\alpha x^2}{2}}$

$\rightarrow f = e^c e^{\frac{\alpha x^2}{2}}$
$\rightarrow f = Ae^{\frac{\alpha x^2}{2}}$

If a wavefunction ψ with energy E

$$means\ \boxed{H\psi = E\psi}$$

Then
$\psi_+ = (a_+\psi)$
$\psi_- = (a_-\psi)$

Also solves the schrodinger eq but with energy
$(E \pm \hbar\omega)$

Energy of a particle is positive
$\psi_{solution}$
$H\psi_{solution} = E_{solution}\psi_{solution}$
$a_-\psi_{solution} \rightarrow E_{new} = E_{sol} - \hbar\omega$
$a_-a_-a_{\psi_{sol}} \rightarrow E_{new} = E_{sol} - 3\hbar\omega$
Since there is no negative energy, there must be some wavefunction/state such that if you apply a_- on it → give zero
ψ_0 =ground state
$\boxed{a_-\psi_0 = 0} \rightarrow solve\ for\ this\ \psi_0$
$\frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx}\psi_0 - im\omega x\psi_0\right) = 0 \rightarrow \frac{\hbar}{i}\frac{d}{dx}\psi_0 = im\omega x\psi_0$
$\boxed{\frac{d}{dx}\psi_0 = -\frac{m\omega}{\hbar}x\psi_0}$

$\boxed{\psi_0 = A\ e^{-\frac{m\omega}{2\hbar}x^2}}$

$\int_{-\infty}^{\infty}\psi_0\psi_0^* dx = |A|^2\int e^{-\frac{m\omega}{\hbar}x^2}dx$
Value of A is
$A = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$

How do we calculate
ψ_1 = first excited state
$\psi_1 = a_+\psi_0$
$\frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx}\psi_0 + im\omega x\psi_0\right) = \psi_1$
$\psi_1 = \frac{1}{\sqrt{2m}}(\frac{\hbar}{i}A\ e^{-\frac{m\omega}{\hbar}x^2} + im\omega xAe^{-\frac{m\omega}{\hbar}x^2}$
$\psi_1 = \frac{1}{\sqrt{2m}}e^{-\frac{m\omega}{\hbar}x^2}(\#x)$
If you calculate the energy of the ground state
$H\psi_0 = \frac{\hbar\omega}{2}\psi_0$
$\frac{\hbar\omega}{2} = energy\ of\ \psi_0$
$\psi_0 = Ae^{-\frac{m\omega}{\hbar}x^2}$
$\boxed{E_m = \left(m + \frac{1}{2}\right)\hbar\omega}$

Summary

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14:07

Schrodinger eq

For a particle of mass m under the influence of a potential V(x) $\left(f = -\frac{dV}{dx}\right)$

$$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x}\ \psi(x,t) + V(x)\psi(x,t) = i\hbar\frac{\delta}{\delta t}\ \psi(x,t) \rightarrow you\ get$$

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi$$

By substituting in $\psi(x,t) = e^{-\frac{i}{\hbar}Et}$

$$\psi = \begin{cases} complex\ in\ most \\ of\ our\ examples \\ \psi(x,t) = e^{-\frac{i}{\hbar}Et} \end{cases}$$

Wavefunction

$$\Psi(x,t) or\ \psi(x) = \begin{cases} differentiable \\ single\ valued \\ \int_{-\infty}^{\infty}\psi\psi^* dx = 1 \rightarrow \text{"}probabilities\text{"} \end{cases}$$

$\Psi(x,t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathrm{E\ t}} \circledast (\mathrm{x})$

$$\psi(x,t) = A\exp -\frac{i}{\hbar}Et\exp -\lambda(x-a)^2$$

$$|\psi|^2 = \psi\psi^* = |A|^2\exp -2\lambda(x-a)^2$$

We computed

$$< x > \ = \int_{-\infty}^{\infty}\psi(x,t)x\psi^*(x,t)dx\,, < p > \ = \int_{-\infty}^{\infty}\psi\frac{\hbar}{i}\frac{d}{dx}\psi^* = 0$$

$$< x^2 > \ = \int_{-\infty}^{\infty}\psi x^2\psi^* dx\,, \qquad < p^2 > \ = \int_{-\infty}^{\infty}\psi\left(\frac{\hbar}{i}\right)^2\frac{d^2}{dx^2}\psi^* dx$$

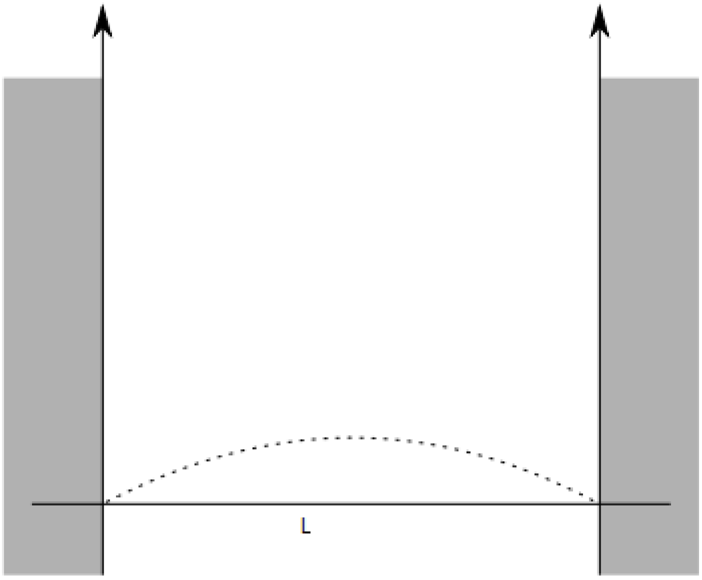
$$< x^2 > \sim\frac{1}{\lambda} \qquad < p^2 > \ \sim\lambda$$

$$\Delta x\Delta p\sim\hbar$$

$$\frac{\hbar^2}{2m}\Psi'' = E\Psi$$

$$\Psi = A\cos\omega x + B\sin\omega x$$

$$-\omega^2 = \frac{2mE}{\hbar^2}$$



Energy is quantized
Solving the problem→

$$E_k = \left(\frac{\hbar^2\pi^2}{2mL^2}\right)\kappa^2$$

$$\psi(x,t) = \sum_{k=1}^{\infty} c_k \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) e^{-\frac{i}{\hbar}E_k t}$$

$|c_k|^2$ =probability of finding the system in the state of E_k
Example

$$\psi(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) e^{-\frac{i}{\hbar}E_{10} t} * \frac{1}{4} + \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) e^{-\frac{i}{\hbar}E_{28} t} * \sqrt{\frac{15}{16}}$$

$$P(E_{10}) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$P(E_{28}) = \frac{15}{16}$$

As soon as one state is measured, wavefunction collapses

$$V = \frac{kx^2}{z} = \frac{m\omega^2}{2}x^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$-\frac{\hbar^2}{2m}\psi'' + \frac{m\omega^2}{2}x^2\psi(x) = E\psi(x)$$

→special function

$$a_\pm = \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} \pm im\omega x\right)$$

$$[a_-, a_+] = \hbar\omega, H = \begin{cases} a_+a_- + \frac{\hbar\omega}{2} \\ a_-a_+ - \frac{\hbar\omega}{2} \end{cases}$$

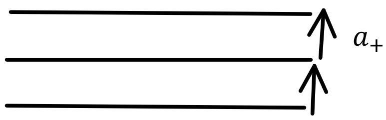
$[a_\pm, H] = \mp\hbar\omega a_\pm$
ψ_* such $H\psi_* = E_*\psi_*$

$$H(a_\pm\psi_*) = (E_* \pm \hbar\omega)(a_\pm\psi_*)$$

$$\psi_2 = a_+ a_+ \psi_0 = a_+\psi_1, E_2 = \frac{5}{2}\hbar\omega$$

$$\psi_1 = a_+\psi_0, E_1 = \left(\frac{\hbar\omega}{2} + \hbar\omega\right) = \frac{3}{2}\hbar\omega$$

ψ_0



$$\boxed{a_-\psi_0 = 0} \rightarrow \boxed{\psi_0 = Ae^{-\frac{m\omega}{2\hbar}x^2}}$$

$$E_0 = \frac{\hbar\omega}{2}$$

SS

13 November 2011
12:25

$$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\psi(x,t) + V_x\psi(x,t) = i\hbar\frac{\delta}{\delta x}\psi(x,t)$$

Suppose that

$$\psi(x,t) = e^{-\frac{iE}{\hbar}t}\psi(x)$$

-stationary state

Can be put into schrodinger eq

$\psi^*(x,t) \times \psi(x,t) = |\psi(x)|^2$ →probability of the system to be between (x,x+dx)

What we need to impose on a wavefunction?

1. Solves schrodinger eq
   $$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\psi(x,t) + V_x\psi(x,t) = i\hbar\frac{\delta}{\delta x}\psi(x,t)$$
2. It has to be differentiable
3. Simple valued
4. Normalizable $\int_{-\infty}^{\infty} dx|\psi(x)|^2 = 1$

$$\int_{-\infty}^{\infty} e^{-\lambda x^2}dx = \sqrt{\frac{\pi}{\lambda}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2}dx = \sqrt{\frac{\pi}{2\lambda^3}}$$

$$< x > \; = \int_{-\infty}^{\infty} \psi^*(x,t)x\,\psi(x,t)dx$$

$$< x^2 > \; = \int_{-\infty}^{\infty} \psi^* x^2 \psi$$

$$(\Delta x) = \sqrt{< x^2 > - < x >^2}$$

Momentum operator

$$\hat{p} = \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta \mathrm{x}}$$

$$\overline{\nabla} f = \frac{\delta f}{\delta x}\hat{\imath} + \frac{\delta f}{\delta y}\hat{\jmath}$$

$$\hat{p}\psi(x,t) = \frac{\hbar}{i}\frac{\delta}{\delta x}\psi(x,t)$$

$$\hat{p}^2\psi(x,t) = \hat{p}\hat{p}\psi$$

$$\frac{\hbar}{i}\frac{\delta}{\delta x}\left(\frac{\hbar}{i}\frac{\delta}{\delta x}\psi\right) = -\hbar^2\frac{\delta^2}{\delta x^2}\psi$$

$$\psi(x,t) = \left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}} e^{-\frac{i}{\hbar}Et} e^{-\lambda x^2}$$

$$\Delta p = \sqrt{< p^2 > - < p >^2} = \hbar\sqrt{\lambda}$$

$$\Delta x = \frac{1}{2\sqrt{x}}$$

$$\Delta p = \hbar\sqrt{\lambda}$$

$\lambda \to \infty \; \Delta p \to \infty$
$\lambda \to 0 \;\; \Delta p \to 0$

$$\Delta x \Delta p \sim \frac{\hbar}{2}$$

Uncertainty principle

Angular momentum (spin)

16 November 2011
12:03

Treatment quite similar to the oscillator a_+ , a_-

Discuss something
$x' = \gamma(x - vt)$
$t' = \gamma\left(t - \frac{vx}{\Omega}\right)$
$y' = y$
$z' = z$

Lorentz transformations

$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\psi(x,t) + V(x)\psi(x,t) = i\hbar\frac{\delta}{\delta t}\psi(x,t)$
Not invariant under Lorentz transformation
Treats time and space differently
$\frac{\delta^2}{\delta x^2}f(x,y) = \frac{1}{c^2}\frac{\delta^2}{\delta t^2}f(x,t)$
Is relativistic invariant
Treats time and space the same

$\psi \sim e^{-\lambda x^2}$
$\rightarrow \Delta x \sim \frac{1}{\lambda}$
$\Delta p \sim \lambda$
$\rightarrow \Delta x\ \Delta p \sim \hbar$
Heisenberg uncertainty principle

Suppose that you consider two operators
$\underline{\theta_1}, \underline{\theta_2}$
These two operators $\theta_1\theta_2$ can be measured simultaneously with any precision if and only if $[\theta_1, \theta_2] = 0$

You <u>cannot</u> measure simultaneously
$\hat{x}, \hat{p} \rightarrow \boxed{[x,p] = i\hbar}$
To measure simultaneously
$\left.\begin{matrix}\theta_1\\ \theta_2\end{matrix}\right\}[\theta_1, \theta_2] = 0$
$x \rightarrow$ arbitrary precision $\Delta x \sim 0$
Often some time passes you measure
$p\ \Delta p \sim 0$
This is ok in quantum mechanics
Why is this useful?
In quantum mechanics one describes a system, by giving the values of a set of operators, that commute with each other(complete set of commuting operators)

$\Delta x \Delta p_x \geq \hbar$
$\Delta y \Delta p_y \geq \hbar$
$\Delta z \Delta p_z \geq \hbar$

$\Delta x \Delta p_y!\,(not!) \geq \hbar$
You can measure x and P_y with all the precision you want

$[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$
$[x, p_y] = [x, p_z] = [y, p_x] = \cdots = [z, p_y] = 0$

You can measure these with all precision

$\bar{L} = \bar{R} \times \bar{p}$
$\bar{R} = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})$
$\bar{P} = (p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k})$

$$\bar{R} \times \bar{p} = \begin{vmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$L_x = yp_z - zp_y$
$L_y = zp_x - xp_z$
$L_z = xp_y - yp_x$

$L_x * L_y \neq L_y * L_x$

<u>Last time</u>

θ_1 and θ_2 are two operators $[x_i, p_x, p_y, p_x^2, \hat{H}, etc]$
You can simultaneously measure θ_1 and θ_2 with all precision if

$[\theta_1, \theta_2] = \theta_1\theta_2 - \theta_2\theta_1 \equiv 0$

Some commutation relations

$$\begin{cases} [x, p_x] = [y, p_y] = [z, p_z] = i\hbar \\ [x, y] = [x, z] = [y, z] = 0 \\ [p_x, p_y] = \cdots = 0 \\ [x, p_y] = [x, p_z] = \cdots = 0 \end{cases}$$

$[AB, C] = A[B, C] + [A, C]B$
$AB.C - C.AB = A.(BC - CB) + (AC - CA)B$

$ABC - ACB + ACB - CAB = ABC - CAB$

$$\boxed{[AB, CD] = AC[B, D] + A[B, C].D + C[A, D]B + [A, C]D.B}$$

Good observables commute with each other such that [A,B]=AB-BA=0

Commutation

21 November 2011
14:17

We discuss today are the

$[L_x, L_z]$
$[L_x, L_z]$
$[L_y, L_z]$
$=?$

Will show that they do not commute

$[L_x, L_y] = [yp_z - xp_y, zp_x - xp_z] = [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]$

$[AB, CD] = AC[B, D] + A[B, C].D + C[A, D]B + [A, C]D.B$

$[yp_z, zp_x] = yz[p_z, p_x] + y[p_z, z].p_x + z[y, p_x]p_z + [y, z]p_z.p_x$

$yz[p_z, p_x] = 0$
$[y, z]p_z.p_x = 0$
$z[y, p_x]p_z = 0$
$y[p_z, z].p_x = -i\hbar\, yp_x$

$[yp_z, zp_x] = -i\hbar$

$[zp_y, xp_z] = zx[p_y, p_z] + z[p_y, x]p_z + z[z, p_x]p_y + [z, x]p_x p_z$

$[zp_y, xp_z] = i\hbar xp_y$

$-[yp_z, xp_z] - [zp_y, zp_x] = 0$

$[L_x, L_y] = i\hbar(xp_y - yp_x)$
$= i\hbar L_z$

$[L_x, L_y] = i\hbar L_z$
$[L_y, L_z] = i\hbar L_x$
$[L_z, L_x] = i\hbar L_y$

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[S_x, S_y] = S_x S_y - S_y S_x = \frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = i\hbar S_z$

$[S_y, S_z] = S_y S_z - S_z S_y = \frac{\hbar^2}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = i\hbar S_x$

$[S_z, S_x] = S_z S_x - S_x S_z = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = i\hbar S_y$

Algebra of SU(2)
Pauli 1929

Angular momentum

23 November 2011
12:09

$[AB,C] = A[B,C] + [A,C]B$
$L_x = yp_z - zp_y$
$L_y = zp_x - xp_z$
$L_z = xp_y - yp_x$

Used $[i, p_i] = i\hbar$

Where i=x,y,z

$[L_x, L_y] = i\hbar L_z$
$[L_y, L_z] = i\hbar L_x$
$[L_z, L_x] = i\hbar L_y$

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[S_x, S_y] = i\hbar S_z$
$[S_y, S_z] = i\hbar S_x$
$[S_z, S_x] = i\hbar S_y$

Notation: sometimes to denote angular momentum, people use J,L,S

S=spin (angular momentum)
L=orbital (angular momentum)
J=total angular momentum
$\boxed{\bar{J} = \bar{L} + \bar{S}}$
All satisfy above notation

We said that two operators that do NOT commute an NOT be measured simultaneously with all precision. There is an "uncertainty principle" for any two operators that do NOT commute
$[\theta_1, \theta_2] \neq 0\# \Rightarrow \Delta\theta_1\Delta\theta_2 \geq \hbar\#$
$[x, p_x] = i\hbar \Rightarrow [\Delta x \Delta p_x \geq \hbar]$
The "algebra" of angular momentum (commutation relation) tells us that L_x, L_y, L_z are NOT a good set of observables
Let us first introduce $\bar{L}^2 = L_x^2 + L_y^2 + L_z^2$

$\bar{L}^2 \equiv \underline{L}^2$

Let us study

$[L^2, L_i]$

i=x,y,z

$[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z]$

$[L_x^2, L_z] = L_x[L_x, L_z] + [L_x, L_z]L_x = L_x i\hbar L_y + i\hbar L_y L_x = -i\hbar(L_y L_x + L_x L_y)$
$[L_y^2, L_z] = i\hbar(L_x L_y + L_y L_x)$
$[L_z^2, L_z] = L_z[L_z, L_z] + [L_z, L_z]L_z = 0$

$[L^2, L_z] = -i\hbar(L_y L_x + L_x L_y) + i\hbar(L_x L_y + L_y L_x) + 0 = 0$

$[L^2, L_i] = 0$

This tells us that two good observables two good operators measure one
$\begin{vmatrix} L^2 \text{ and } L_z \\ L^2 \text{ and } L_y \\ L^2 \text{ and } L_x \end{vmatrix}$
In all the books, people choose

$L^2 \; and \; L_z$

Exactly as we did in the oscillator, we will define two new operators

$L_+ = L_x + iL_y$

$L_- = L_x - iL_y$

$[L_z, L_+] = [L_z, L_x + iL_y] = [L_z, L_x] + i[L_z, L_y] = \hbar(L_x + iL_y) = \hbar L_+$

$[L_z, L_-] =$

$[L^2, L_+] = [L^2, L_x + iL_y] = [L^2, L_x] + [L^2, L_y] = 0$

$[L^2, L_-] = 0$

$[J_x, J_y] = i\hbar J_z$

etc

$[L_y, L_z] = i\hbar L_x$

$[L_z, L_x] = i\hbar L_y$

Convention: Use $J_z \; and \; J^2$ as observables

$J_\pm = J_x \pm iJ_y$

$J^2 = J_x^2 + J_y^2 + J_z^2$

And

J_z

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\psi_{l,m}$

$l \to J^2, m \to J_z$

$J^2\psi_{l,m} = \hbar^2 l(l+1)\psi_{l,m}$

$J_z\psi_{l,m} = \hbar m\psi_{l,m}$

$m: -l, \dots, l$

Limits of (1)

Quantum numbers l and m

Consider a particle of spin 1/2 ≡ electron, proton, neutron, quark

$$\psi_{l,m}; \quad l = \frac{1}{2}, \quad m = -\frac{1}{2}, \frac{1}{2}$$

Two states

$\psi_{\frac{1}{2},\frac{1}{2}}$

And

$\psi_{\frac{1}{2},-\frac{1}{2}}$

Spin 1-> photons

$\psi_{1,1}$

$\psi_{1,0}$

$\psi_{1,-1}$

Spin 2 -> gravitons

$\psi_{2,2}$

$\psi_{2,1}$
$\psi_{2,0}$
$\psi_{2,-1}$
$\psi_{2,-2}$

$J_{\pm} = ladder\ operators$

Change between states for m

$$J_+ = J_x + iJ_y = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$J_- = J_x - iJ_y = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$J^2 = J_x^2 + J_y^2 + J_z^2 = \left[\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]^2 + \left[\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]^2 + \left[\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]^2 = \cdots = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

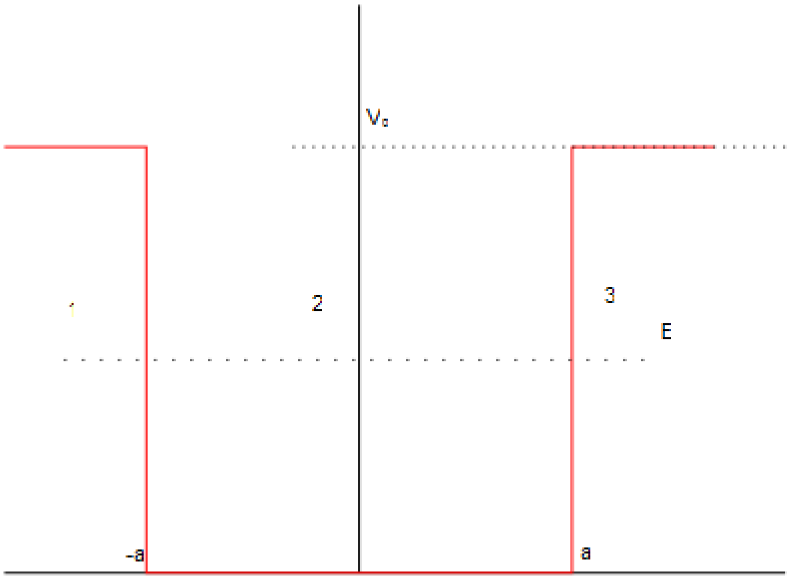
$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$J_z v_1 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$J_z v_2 = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$J^2 v_1 = \frac{3}{4}\hbar^2 v_1$$

$$J^2 v_2 = \frac{3}{4}\hbar^2 v_2$$

$$J_+ v_1 = \hbar.0$$
$$J_+ v_2 = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar v_1$$
$$J_- v_1 = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$J_- v_2 = 0$$



Schrodinger eq in region 1 (left) and 3 (right)

$$-\frac{\hbar^2}{2m}\psi''(x) + V_0\psi(x) = E\psi_0$$

$$\rightarrow \boxed{\psi'' + \frac{2m}{\hbar^2}(E - V_0)\psi(x) = 0}$$

In 2

$$-\frac{\hbar}{2m}\psi'' = E\psi$$

$$\rightarrow \boxed{\psi'' + \frac{2m}{\hbar^2}\psi(x) = 0}$$

$$\psi'' + \omega^2\psi = 0 \rightarrow A\cos\omega x + B\sin\omega x$$
$$\psi'' - \omega^2\psi 0 =\rightarrow A\, e^{+\omega x} + Be^{-\omega x}$$

In regions 1 and 3

$$\psi'' + \frac{2m}{\hbar^2}(E - V_0)\psi = 0$$
$$\psi'' - \omega^2\psi$$

$$\boxed{\omega^2 = \frac{2m}{\hbar^2}(V_0 - E)}$$

In 1

$$\psi_1 = Ae^{\omega_1 x} + Be^{-\omega_1 x}$$

In 3

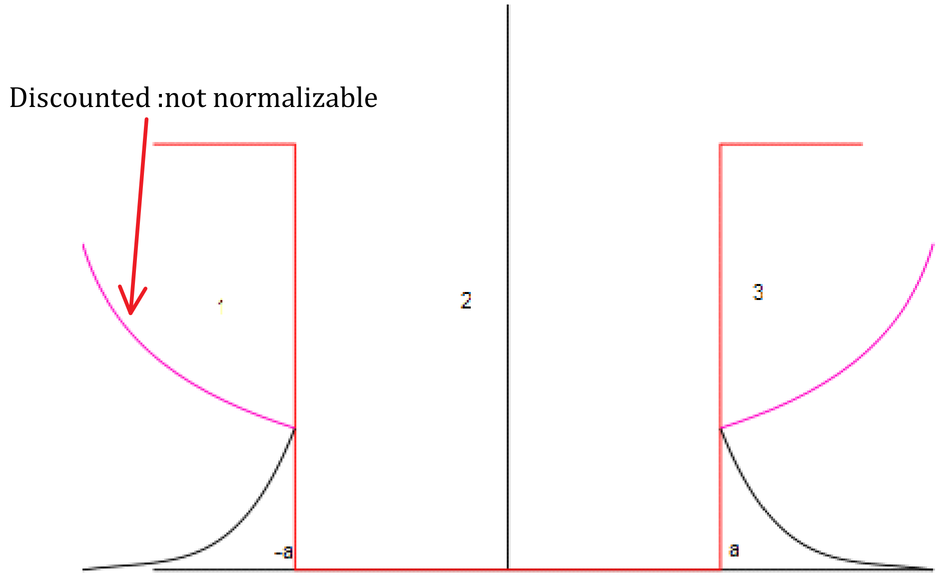
$$\psi_3 = Ce^{\omega_1 x} + De^{-\omega_1 x}$$

In 2

$$\psi'' + \omega_2^2\psi = 0$$
$$\omega_2^2 = \frac{2m}{\hbar^2}E$$

$$\boxed{\psi_2 = Q\cos\omega_2 x + F\sin\omega_2 x}$$

Discounted :not normalizable



Schrodinger eq in 1, in 3

In region 1

$$\psi_1 = Ae^{\omega_1 x}$$

In region 2

$$\psi_2 = Q\cos\omega_2 x + F\sin\omega_2 x$$

In region 3

$$\psi_3 = De^{-\omega_1 x}$$

$$\omega_1^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

$$\omega_2^2 = \frac{2m}{\hbar^2}E$$

A wave function needs to be continuous and differentiable

i. $\psi_1(x=-a) = \psi_2(x=-a)$
ii. $\psi_1'(x=-a) = \psi_2'(x=-a)$
iii. $\psi_2(x=a) = \psi_3(x=a)$
iv. $\psi_2'(x=a) = \psi_3'(x=a)$

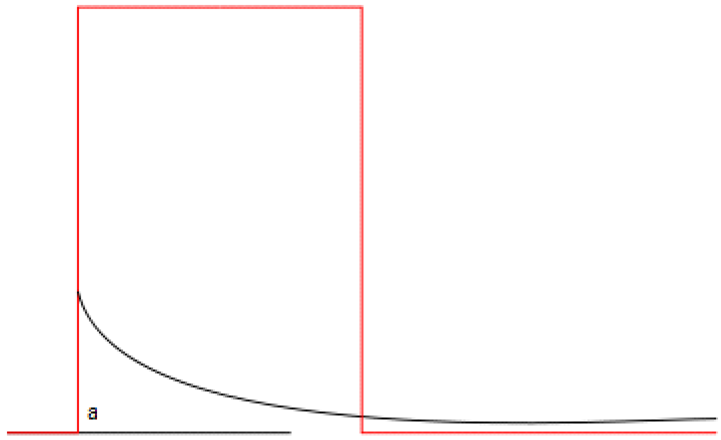
i) $Ae^{-\omega_1 a} = Q\cos\omega_2 a - F$
ii) $A\omega_1 e^{-\omega_1 a} = Q\, sin\,\omega_2 a\ + F\, cos\,\omega_2 a$
iii) $\omega_2(Q\cos\omega_2 x + F\sin\omega_2 x) = De^{-\omega_1 a}$
iv) $\omega_2(Q\cos\omega_2 x + F\sin\omega_2 x) = -D\omega_1 e^{\omega_1 a}$

A,Q,F,D

Unknowns

4 equations: put in mathematica

$$\int_{-\infty}^{-a}\psi\psi^* + \int_{-a}^{a}\psi\psi^* + \int_{a}^{\infty}\psi\psi^* = 1$$



Quantum tunnelling

Lagrangian → definition

$L = T - V$

where

$T = \frac{m\dot{x}^2}{2}$

V=potential

1. Free Particle → $\boxed{V = 0}$ → $\boxed{L = \frac{m\dot{x}^2}{2}}$
2. Oscillator

$\rightarrow V = \frac{Kx^2}{2} \rightarrow \boxed{L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}}$

F=-kx

Action,

$S = \int_{t_0}^{t} L\, dt$

````

Momentum action principle

impose→ minimize the action $\boxed{\delta S = 0}$

→ find some eqs Euler Lagrange eps

$\boxed{\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right) = \frac{\delta L}{\delta x}}$

$\frac{\delta L}{\delta \dot{x}} = \begin{matrix} \rightarrow Free & m\dot{x} \\ \rightarrow Oscillator & m\dot{x} \end{matrix} \rightarrow \frac{d}{dt}(m\dot{x}) = m\ddot{x} + \dot{m}\dot{x}$

$\frac{\delta L}{\delta x} = \begin{matrix} \rightarrow Free & 0 \\ \rightarrow Oscillator & -kx \end{matrix}$

$Free\ particle \quad m\ddot{x} = 0$

$Oscillator \quad m\ddot{x} = -kx$

Much more complicated if mass ≠ constant

Interference + diffraction

$\psi = \psi_1 + \psi_2$

$P = |\psi_1|^2 + |\psi_2|^2 + 2\psi_1\psi_2$

$\psi_1\psi_2^* + \psi_2\psi_1^*$

$P = P_1 + P_2 + Interference\ term$
````

$L = T - V = \frac{m\dot{x}^2}{2} - V(x)$

$\boxed{\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right) = \frac{\delta L}{\delta x}}$

Action

$$S = \int_{t_0}^{t_t} L\, dt$$

2 slit

$\psi_A = \psi_1 + \psi_2$

$I = |\psi_a|^2 = |\psi_1 + \psi_2|^2 = \psi_1^2 + \psi_2^2 + 2\psi_1\psi_2$

$2\psi_1\psi_2 = interference$

3 slit

$\psi_A = \psi_1 + \psi_2 + \psi_3$

$I = |\psi_1 + \psi_2 + \psi_3|^2$

4 slit

$\psi_A = \psi_1 + \psi_2 + \psi_3 + \psi_4$

$I = |\psi_1 + \psi_2 + \psi_3 + \psi_4|^2$

Electrons

$\psi_A = \psi_1 + \psi_2$

Wavefunction at x=A

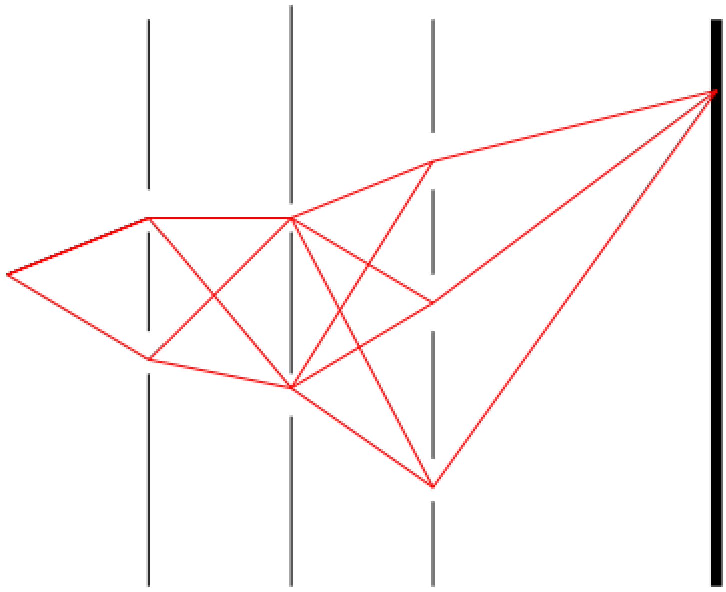
Prob (electron at A) = $|\psi_A|^2$

$= |\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + \psi_1\psi_2^* + \psi_2\psi_1^*$

$\psi_A = \sum \psi_i(x_i)$

$\boxed{P(A) = |\sum \psi_i|^2}$

ψ_A =sum over all paths to get to the point A



$\psi_A = \sum \psi_{ijk}$

$\psi(x_1 t_1, x_2 t_2)$ =sum all possible paths/ways that go from $(x_1 t_1) \rightarrow (x_2 t_2)$
ψ_i

$$Amplitude(x_1, t_1, x_2, t_2) = \int DX \exp -\frac{i}{\hbar}\int_{t_1}^{t_2} L\, dt$$

DX means sum over ALL paths

Equivalent to schrodinger eq for $L = \frac{m\dot{x}^2}{2} - V(x)$

1. Classical limit $\hbar \rightarrow 0$

   $\Delta x \Delta p \geq \hbar$

   $\exp -\frac{i}{\hbar}\int_{t_1}^{t_2} L\, dt \rightarrow sin + cos$

   $A \cong \exp -\frac{i}{\hbar} S_{classical}$

   $S \rightarrow path$

All this course we worked with NON relativistic quantum mechanics

Exam

Mon Jan 16 2pm
Email carlos if you want to redo any of the mid terms

Schrodinger eq will be written in exam- don't need to memorize
Given a $\psi(x,t) \rightarrow$

$$< x >$$
$$< x^2 >$$
$$< p >$$
$$< p^2 >$$
$$\Delta x \, \Delta p \geq \hbar$$

Operators

$|\theta_1, \theta_2 \dots, \theta_n|$

Good operators if

$[\theta_1, \theta_2] \Rightarrow all\ prescision\ etc$

Good observables
(L_x, L_z)
(p_x, p_y)

Angular momentum

$L_i J_i etc$
$L_x = yp_z - zp_y\ etc$
$L^2 = L_i^2$
Where i=x,y,z
$[L_x, L_y] = i\hbar L_z$
etc
$L_\pm = effects\ on\ the\ states$

$$\begin{pmatrix}1\\0\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix}$$
$$J_+ \begin{pmatrix}1\\0\end{pmatrix} = 0$$
$$J_- \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$
$$J_+ \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$
$$J_- \begin{pmatrix}0\\1\end{pmatrix} = 0$$
$$J_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}; J_y = \begin{pmatrix}0 & i\\-i & 0\end{pmatrix}$$
$$\boxed{J_z = \begin{pmatrix}1 & \\ & -1\end{pmatrix}}$$
$$(J_z, J^2)$$

Oscillator

$$-\frac{\hbar}{2m}\frac{\delta^2}{\delta x^2}\psi + V\psi = i\hbar\frac{\delta}{\delta t}\psi$$
$$V = \frac{kx^2}{2}$$
$$a_+ = \frac{1}{\sqrt{2}m}\left(\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right)$$

$$a_- = \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right)$$

$[a_+, a] = ?$

$[a_\pm, H] = ?$

Spectrum discrete

$$E_m = \left(n + \frac{1}{2}\right)\hbar\omega$$

$\psi \Rightarrow \boxed{a_-\psi_0 = 0} \rightarrow solved$

$\psi_0 = e^{-x^2}$

$\psi_1 = a_+\psi_0$

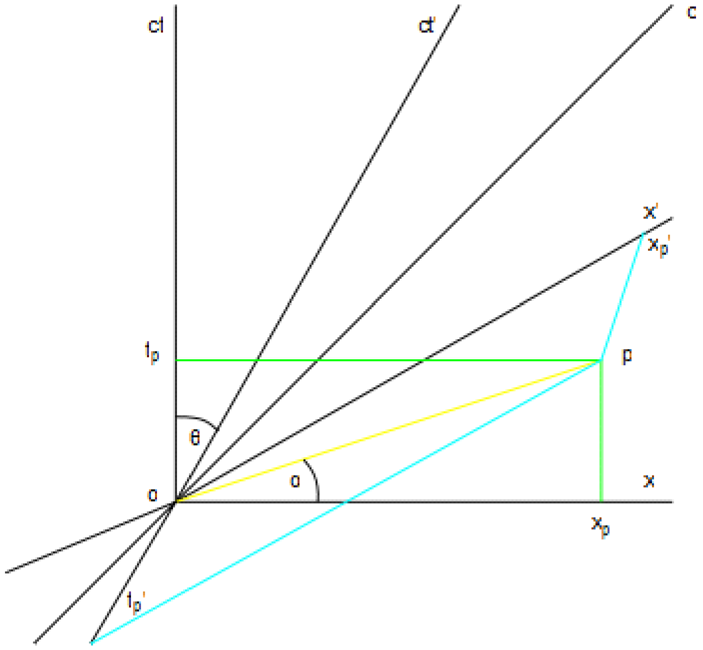
$\psi_2 = a_+a_+\psi_0$

Etc

Go over these things to ensure understanding

FTL

27 November 2011
15:12



0p is a faster than light path
In frame S, op is forwards in time $t_p > t_0$
In frame S', 0P is backwards in time, $t_p' < t_0'$
⇒ If a faster than light signal is possible, Then in some frames it is forwards in time, but in other frames it is backwards in time

$\tan\theta = \frac{v}{c}$

$\tan\alpha = \frac{c}{u}$

Where u is the speed of the tachyon
Backwards in time in S' if the relativity velocity of the frames satisfies

$$\tan\theta > \tan\alpha$$

$$\Rightarrow \frac{v}{c} > \frac{c}{u} \Rightarrow v > \frac{c^2}{u}$$

Directly from Lorentz transformation
For 0p,

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

$$x' = \gamma\left(x - \frac{v}{c}ct\right)$$

Speed of tachyon $u = \frac{x}{t}$

$t' < 0\ if\ ct - \frac{v}{c}x < 0$

$$\Rightarrow ct\left(1 - \frac{v}{c^2}\frac{x}{t}\right) < 0$$

$$\left(1 - \frac{v}{c^2}\frac{x}{t}\right) \Rightarrow 1 - \frac{v}{c^2}u$$

$$\Rightarrow v > \frac{c^2}{u}$$

So far, this is not a terminal problem
The problem is that once we accept faster then light/backwards in time in some frame is possible, Postulate I says that it is possible in all inertial frames in particular, it implies that if 0p is possible, then so is motion PQ

Closed path 0PQ is incompatible with causality (Grandfather paradox)

Problem Sheets

21 November 2011
15:18

Sheet 3

1. Track $= 1.05 * 10^{-9} m$
   Speed $= 0.992 * 3 * 10^8\ ms^{-1}$
   ⇒ lifetime in LAB FRAME
   $$= \frac{1.05 * 10^{-9}}{0.992 * 3 * 10^8} = 3.53 * 10^{-18} sec$$
   Lifetime in REST FRAME
   $$= \frac{1}{\gamma}(3.53 * 10^{-18}) = 0.445 * 10^{-18} sec$$
   $$S^2 = -c^2t^2 + x^2 \text{ lab}$$
   $$= -c^2t'^2 \text{ rest}$$
   $$\Rightarrow t'^2 = t^2 - \frac{x^2}{c^2}$$
   $$\Rightarrow t' = t\sqrt{1 - \frac{v^2}{c^2}}$$
   $$= \gamma^{-1} t$$
2. Lifetime in rest frame $= 26 * 10^{-9} s$
   At 0.99c
   $$\gamma = \frac{1}{\sqrt{1 - 0.99^2}} = 7.1$$
   ⇒ lifetime in lab/earth frame $= \gamma(26 * 10^{-9})$
   $= 184 * 10^{-9} s$
   Distance $= 1.84 * 10^{-7} * 0.99 * 3 * 10^8 m$
   $= 54.6 m$
   $$1 - \frac{v^2}{c^2} = \left(1 + \frac{v}{c}\right)\left(1 - \frac{v}{c}\right)$$
   $$1 - 0.99^2 = (1.99)(0.01) = 0.02$$
   $$\gamma = \frac{1}{\sqrt{\square}} = \frac{1}{\sqrt{0.02}} = \frac{10}{\sqrt{2}} \approx \frac{10}{1.4} = 7$$
3.
   i. Earth time when astronaut reaches VEGA
   $$= \frac{26}{0.99} yrs = 26 * 26\ yrs$$
   $$\frac{1}{0.99} = \frac{1}{1 - 0.01} = (1 - 0.01)^{-1}$$
   $$= 1 + 0.01 + 0(0.01)^2 = 1.01$$
   ii. Time to receive radio signal=26.26+26=52.26 yrs
   iii. Astronaut time at VEGA= $\frac{1}{\gamma}\ 26.26 = 3.7\ yrs$
   $$\gamma = \frac{1}{\sqrt{1 - 0.99^2}} = 7.1$$
   $$s^2 = -c^2t'^2$$
   $$= -c^2t^2 + x^2$$
   $$t'^2 = t^2 - \frac{x^2}{c^2} = t^2\left(1 - \frac{v^2}{c^2}\right)$$
   $$t'^2 = (26.26)^2 - 26^2$$
   $$= (26 + 0.26)^2 - 2.6^2 \cong (52 * 0.26) = 13$$
4.
   i. Length earth= $\frac{1}{\gamma}$ plane length= $(1 - 2.2 * 10^{-12}) L_{plane}$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

$$\cong 1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots$$

v=630 m/s

c=3*10^8 m/s

$$\frac{v}{c} = \frac{630}{3*10^8} = 2.1*10^{-6}$$

$$= 1 + 2.2*10^{-12}$$

Change in plane length as measured on earth/length

$$= 2.2*10^{-12} L/L$$

$$= 2.2*10^{-12}$$

Sheet 2

1) 1st half

Invariant spacetime interval

$$S^2 = -c^2t^2 + x^2$$

Earth frame

$$S^2 = -5^2 + 4.9^2$$

Astronaut frame

$$s^2 = -c^2\tau^2$$

$$\Rightarrow \tau^2 = 5^2 - 4.9^2$$

$$= 9.9*0.1$$

$$= 0.99$$

$$\tau = \sqrt{0.99}$$

$$= 0.995$$

Total $= 0.995*2 = 1.99\ yrs$

OR

$$\frac{v}{c} = \frac{4.9}{5} = 0.98$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{0.199}$$

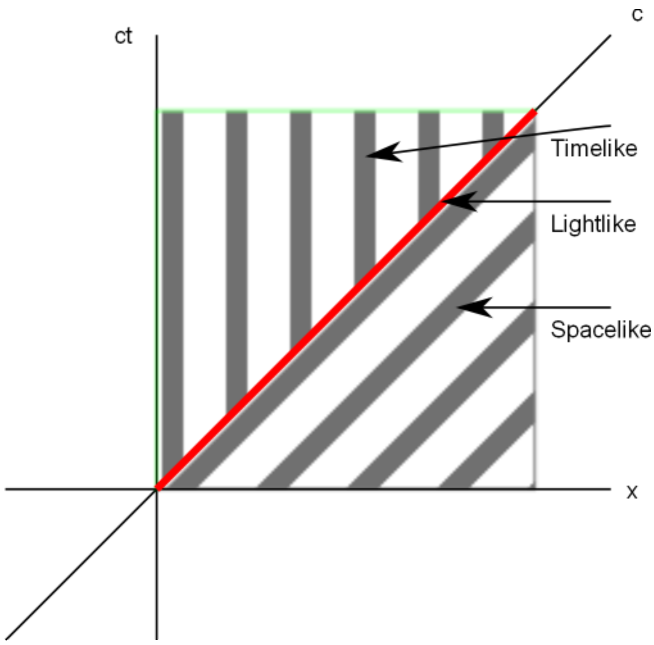
Total time in astronaut frame

$$= \gamma^{-1} * earth\ time$$

$$= 0.199*10 = 1.99\ yrs$$

2) Simultaneity- straight out of lecture notes

3) Timelike, lightlike, spacelike



As long as motion is timelike, motion is always forwards in time

1 Vectors& Newtonian dynamics

04 October 2011
10:11

Position vector

$$\bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Any vector

$$\bar{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Index notation

Scalar product of vectors

$$\bar{u}.\bar{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{\mathrm{i}=1}^{3} u_i v_i$$

Now introduce Kronecker delta symbol

$$\delta_{ij} = \begin{matrix} 1\ if\ i = j \\ 0\ if\ i \neq j \end{matrix}$$

Consider this as a matrix

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}$$

Scalar product is

$$\bar{u}.\bar{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} u_i v_j$$

Matrix form

$$\bar{u}.\bar{v} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

"Einstein sum convention

So just write $u_i v_i$ instead of $\sum_{i=1}^{3} u_i v_i$

Notice that this is what we did with the metric (distance relation)

$ds^2 = dx^2 + dy^2 + dz^2 = \delta_{ij} dx_i dx_j$

^scalar product ^metric

In general, $ds^2 = g_{ij} dx_i dx_j$

Where the metric g_{ij} determines the shape of the space eg 3-sphere, hyperboloid

Flat Euclidean space $g_{ij} = \delta_{ij}$

Rotations

Restrict to 2 dimensions for simplicity

Vector

$$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1$$

Under rotation

$$\bar{v} \to \bar{v}' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = v_1'$$

$$v_i' = R_{ij} v_j = \sum_{j=1}^{2} R_{ij} v_j$$

In matrix notation

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$R_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

R has a special property

$R^T R = 1$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We say that R is "orthogonal"

In 3 dims, R_{ij} where $i_{ij} = 1,2,3\ (3x3\ matrix)$

Still Orthogonal $R^T R = 1$

N dimensions → $\frac{1}{2}N(N-1)$ angles

"Group theory"- mathematics of symmetry

<u>Frame of reference</u>

This is a system of measuring space and time , set up by observers with identical motions

In general, the description of events will be different in different frames, BUT underlying reality is the same

Eg

One frame S which is fixes relative to the earth

Another frame S' could be aboard a plane

Each frame will assign position to an event $S \to (x, y, z)$ and time t $S' \to (x', y', z')$ **and time t**

Newtonian dynamics

Essential feature is the special role given to <u>inertial</u> frames of reference i.e. fixed, or moving with uniform velocity NOT accelerating.

Postulates

1. The laws of dynamics are the same in all inertial frames (all inertial frames are equivalent)
   This is a relativity principle- it means there is no absolute rest frame, only relative motion is important
2. Since all inertial frames are equivalent, the simplest effect of an interaction(force) is to produce an acceleration
   $\to \bar{F} = m\frac{d^2\bar{x}}{dt^2}$
   Forces change acceleration, not velocity
3. Dynamics takes place in flat Euclidean space ⇒ laws of dynamics are invariant under rotations & translations deep theorem (Noether's theorem)
   Translation invariance → momentum is conserved
   Rotation invariance → angular momentum conserved

<u>Galilean transformations</u>

Relate measurements in different inertial frames

(look at 1 space dim for simplicity)

"moving" frame (s' with velocity v) has coords (x',t')

Stationary frame (s) has coords (x,t)

In S, event p has coordinates (x_p, t_p)

Clearly, $x'_p = x_p - vt_p$

Assume $t'_p = t_p$

Applies to any event P, so we have the general relation between coordinates in frames S and S' (galilean transformations)

$$\begin{cases} t' = t \\ x' = x - vt \end{cases}$$

$$in\ 3\ dim \begin{cases} t' = t \\ \bar{x}' = \bar{x}' - \bar{v}'t \end{cases}$$

Invariance of equation of motion

$$\bar{F} = m\frac{d^2\bar{x}}{dt^2}\ \ should\ have\ \ \bar{F} = m\frac{d^2\bar{x}'}{dt'^2}$$

$$\Rightarrow \bar{F} = m\frac{d^2\bar{x}'}{dt'^2} = m\frac{d^2}{dt^2}(\bar{x} - \bar{v}t) = m\frac{d^2\bar{x}}{dt^2}$$

Postulate 1 holds

2 Space and time in special relativity

10 October 2011
10:45

Newtonian picture is changed radically in special relativity (einstein 1905)
Special relativity is based on the following postulates

Postulate 1A

The laws of physics are the same in all inertial frames

Postulate 1B

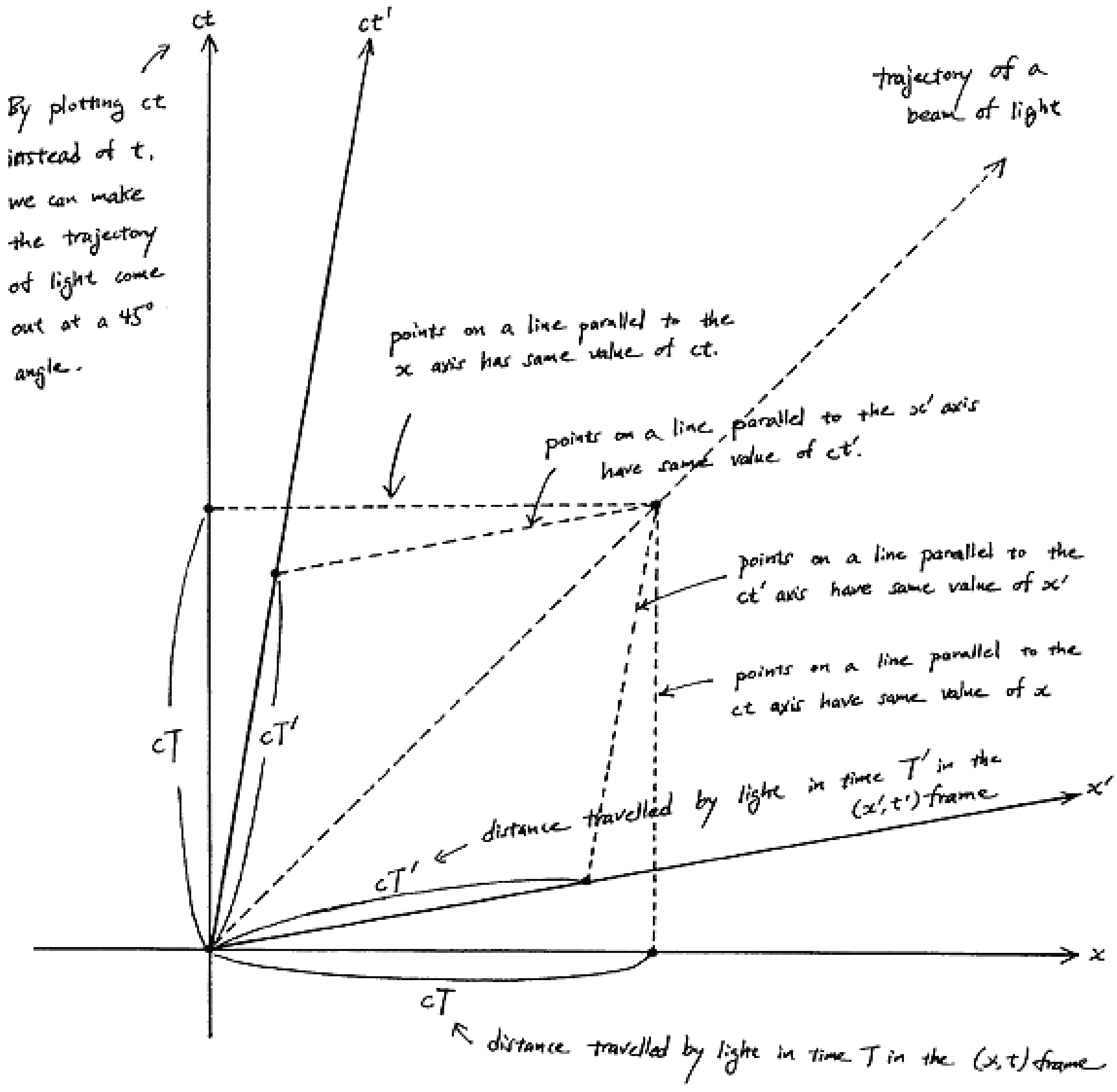
The speed of light is the same in all inertial frames

Note Postulate 1B introduces a new fundamental constant c into physics. (compare with quantum mechanics , scale h)

Postulate 1A extends relativity principle (equivalence of inertial frames) to all of physics in particular including electromagnetism, not just particle dynamics

To satisfy postulate 1B, we need to change our ideas of space and time

Use ct and x as axes- both have dimensions of length



☆ The Galilei Transformation tilts the t' axis.

☆ The Lorentz Transformation tilts both x' and t' axes.

Measure speed of light in new frame S'. To keep speed of light = c, need to change both axes (ct', x') both different from (ct, x)
⇒ time is different in different inertial frames

Lorentz Transformations
These show how space and time coordinates are related in frames S and S' with relative velocity v
[For simplicity, assume $\bar{v}$ is along x-axis]
Postulate 1B

$$x' = \gamma\left(x - \frac{v}{c}ct\right)$$
$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$
*

Check the speed of light

$$Vel\ in\ S = \frac{x}{t} = u$$
$$Vel\ in\ S' = \frac{x'}{t'} = \frac{\gamma\left(x - \frac{v}{c}ct\right)}{\gamma\left(t - \frac{v}{c^2}x\right)} = u'$$
$$\frac{\gamma\left(\frac{x}{t} - v\right)}{\gamma\left(1 - \frac{v}{c^2}\frac{x}{t}\right)}$$
$$\Rightarrow \boxed{u' = \frac{u - v}{1 - \frac{uv}{c^2}}}$$

This formula relates a velocity u measured in S to the velocity u' measured in S'
Note $u' \approx u - v$ only when $u, v \ll c$
Postulate 1B says that if u=c, then u'=c
Set u=c

$$\Rightarrow u' = \frac{c - v}{1 - \frac{cv}{c^2}} = c$$

So Postulate 1B is satisfied by transformations
NB this would be true for any choice of gamma
Postulate 1A
This means we must have the same transformation from S' to S

$$x = \gamma\left(x' + \frac{v}{c}ct'\right)$$
$$ct = \gamma\left(ct' + \frac{v}{c}x'\right)$$
**

NB change sign of relative velocity
Check consistency of * and **

$$x = \gamma^2\left(x - vt + v\left(t - \frac{v}{c^2}x\right)\right) = \gamma^2\left(1 - \frac{v^2}{c^2}\right)x$$
$$ct = \gamma^2\left(ct - \frac{v}{c}x + \frac{v}{c}(x - vt)\right) = \gamma^2\left(1 - \frac{v^2}{c^2}\right)ct$$
$$\Rightarrow \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$
$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

So Postulate 1A shows that the scale factor γ is not arbitrary (as allowed by Postulate 1B alone) but is velocity dependent,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

To summarise

The lorentz transformations are

$x' = \gamma\left(x - \frac{v}{c}ct\right)$

$t' = \gamma\left(t - \frac{v}{c^2}x\right)$

Where

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

NB- transverse space coordinates are unchanged

Note that these reduce to galilean transformations when $c \to \infty$

Minkowski Spacetime

Since Lorentz transformations mix space and time, the geometry relevant for special relativity is 4-dimensional spacetime

However, 4-dim Minkowski spacetime is NOT just Euclidean space

Recall the Lorentz transformations

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

$$x' = \gamma\left(x - \frac{v}{c}ct\right)$$

So both x and t are changed under a change of frame S to S'

But something is left invariant.

Invariant is

$$S^2 = -c^2t^2 + x^2$$

(in 3d $S^2 = -c^2t^2 + x^2 + y^2 + z^2$)

Check

$$\begin{aligned} S'^2 &= -c^2t'^2 + x'^2 \\ &= -\gamma^2\left(ct - \frac{v}{c}x\right)^2 + \gamma^2\left(x - \frac{v}{c}ct\right)^2 \\ &= \gamma^2\left[-c^2t^2 + 2vxt - \frac{v^2}{c^2}x^2 + x^2 - 2vxt + \frac{v^2}{c^2}c^2t^2\right] \\ &= -\gamma^2\left(1 - \frac{v^2}{c^2}\right)c^2t^2 + \gamma^2\left(1 - \frac{v^2}{c^2}\right)x^2 \\ &= -c^2t^2 + x^2 \end{aligned}$$

Since $\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}$

So the combination $S^2 = -c^2t^2 + x^2 + y^2 + z^2$ is always the same, no matter which frame of reference we use

Analogue of distance in 3d Euclidean space

Call s^2 the (square of the) spacetime interval between origin and point

Introduce position 4-vector $x = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$

Generalising 3-vector $x^1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, i = 1,2,3$

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \mu = 0,1,2,3$$

0=time; 1,2,3= space

A Lorentz transformation is $x^\mu \to x'^\mu$

Where

$$\underset{4*1}{x'^\mu} = \underset{4*4}{L^\mu_\text{v}} \; \underset{4*1}{x^\text{v}} \text{ where } L^\mu_v = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

☐ CHECK THIS
SGPC sheet

Generalizes rotations in 3d space

$$\underset{3*1}{x'^i} = \underset{3*3}{R^i_j} \underset{3*1}{x^j}$$

We can write the spacetime interval as

$$S^2 = (ct \quad x \quad y \quad z)\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$= x^\mu g_{\mu v} x^v = x^T gx \text{ in matrix notation}$$

$$Where\ g_{\mu v} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This generalizes 3d distance $S^2 = x^T 1x = x^i \delta_{ij} x^j$

The 4x4 matrix $g_{\mu v}$ specifies the spacetime interval in Minkowski spacetime it is called the <u>metric</u>

Now re-check that S^2 is invariant under Lorentz transformations

$$S^2 = g_{\mu v} x^\mu x^v$$

Lorentz tranf $x^\mu \to x'^\mu = L^\mu_v x^v$

Check

$$S^2 = g_{\mu v} x'^\mu x'^v$$
$$= g_{\mu v} L^\mu_\rho x^\rho L^v_\sigma x^\sigma$$
$$g_{\mu v} x^\mu x^v = g_{\rho\sigma} x^\rho x^\sigma$$

$\Rightarrow S^2$ is invariant if

$$g_{\mu v} L^\mu_\rho L^v_\sigma = g_{\rho\sigma}$$

Matrix notation

$$S^2 = x'^T gx'$$
$$= x^T L^T gLx$$
$$= x^T gx \text{ if } S^2 \text{ is invariant}$$

$$\Rightarrow \boxed{L^T gL = g}$$

CHECK

$L^T gL$

$$= \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma^2\left(1-\frac{v^2}{c^2}\right) & 0 & 0 & 0 \\ 0 & \gamma^2\left(1-\frac{v^2}{c^2}\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g$$

Time in Minkowski spacetime compare 2 paths through Minkowski spacetime

```
ct  |
    |  \
1   ↑  ↑   2
    |  /          x
    +  −  −  −  −  −
```

What does an observer measure as "time"
"proper time" as measured in a frame of reference where the observer is at rest is t where
$s^2 = -c^2t^2$
Infinitesimally,

$$ds^2 = -c^2dt^2$$

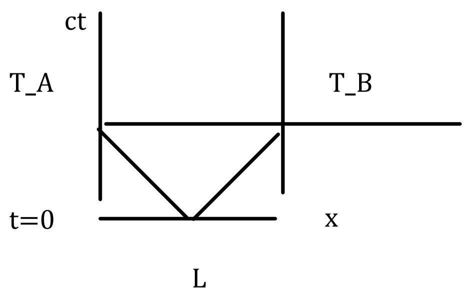
So observer 1 measures time elapsed as t where $\int_{path\ 1} ds$ is the spacetime interval and
$ds^2 = -c^2dt^2$
Similarly for observer 2, there the spacetime interval is $\int_{path\ 2} ds$
But spacetime interval along path 2 is less than path 1
Because $ds^2 = -c^2dt^2 + dx^2 \Rightarrow$ path with biggest value of S is the one where there is no motion in the space direction
This is the famous "astronaut paradox" astronaut 1 stays of earth
Astronaut 2 goes on a fast round trip to alpha centauri and back. When they meet back on earth astronaut 2 is younger then astronaut 1
This is not a paradox because paths 1 and 2 are genuinely different. Path 1 is inertial, path 2 is not. So there is no symmetry between them-> cannot say path 2 is at rest and path 1 moves

3 Measurement of space and time

18 October 2011
10:45

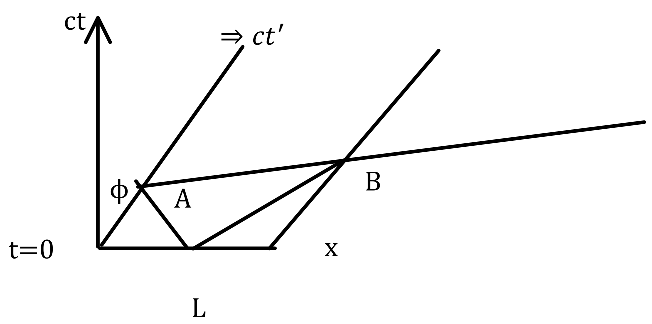
<u>3.1 Simultaneity</u>
Simultaneity is not an absolute property of two events but depends on the frame of reference
Operational definition of simultaneity

Consider a rod of length L
Send a light signal from the midpoint.
this reaches the ends of the rod at the same time (measured in the rest frame of the rod
$\Rightarrow t_A = t_B$



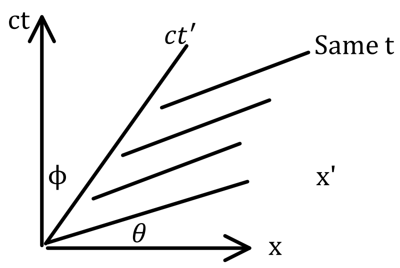
Line of simultaneity (fixed t, parallel to x axis)

Sow suppose rod is moving, with velocity v
Frame S' is co-moving with the rod



Line of simultaneity in frame S' $t'_A = t'_B$
$\Rightarrow x'$

In moving frame S', events A and B are considered simultaneous
Nb: uses both postulates of SR
1. Allows us to use same experimental definition of simultaneity
2. $\Rightarrow$ speed of light is same in all frames



This is why we use skew axes (ct', x') in frame S'
Prove $\theta = \phi$

$$x_A = vt_A = \frac{L}{2} - ct_A \Rightarrow ct_A = \frac{L}{2}\left(\frac{1}{1+\frac{v}{c}}\right)$$

$$x_B = L + vt_B = \frac{L}{2} + ct_B \Rightarrow ct_B = \frac{L}{2}\left(\frac{1}{1-\frac{v}{c}}\right) \qquad (2)$$

$\Rightarrow x_B - x_A = ct_B + ct_A \qquad (1)$

Obviously, $t_A \neq t_B$

However, by definition $t'_A = t'_B$

Angles

$$\tan\phi = \frac{x_A}{ct_A} = \frac{v}{c}$$

$$\tan\theta = \frac{ct_B - ct_A}{x_B - x_A} = \frac{ct_B - ct_A}{ct_B + ct_A}$$

Using (1)

$$= \frac{t_B - t_A}{t_B + t_A}$$

$$= \frac{1 + \frac{v}{c} - \left(1 - \frac{v}{c}\right)}{1 + \frac{v}{c} + \left(1 - \frac{v}{c}\right)}$$

Using (2)

$$= \frac{v}{c}$$

Check using Lorentz transformations

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

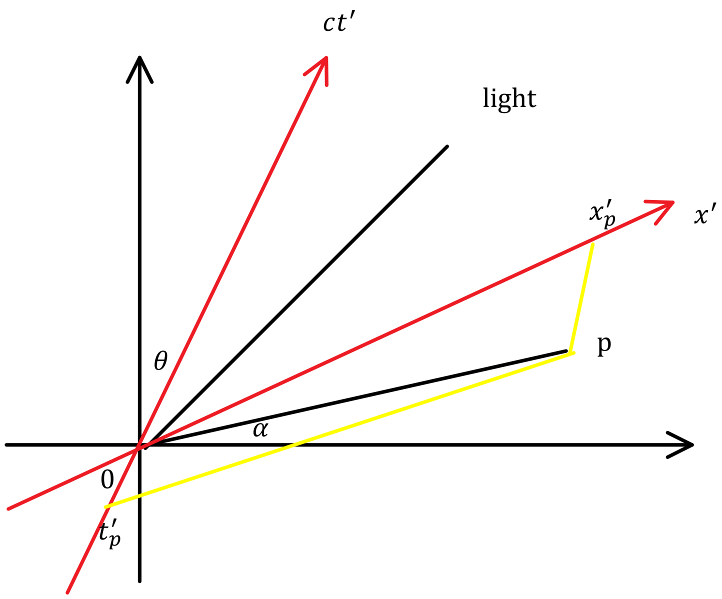
$$x' = \gamma\left(x - \frac{v}{c}ct\right)$$

$$x' \text{ axis is } t' = 0 \Rightarrow ct - \frac{v}{c}x = 0 \Rightarrow \tan\theta = \frac{ct}{x} = \frac{v}{c}$$

$$t' \text{ axis is } x' = 0 \Rightarrow x - \frac{v}{c}ct = 0 \Rightarrow \tan\phi = \frac{x}{ct} = \frac{v}{c}$$

3.2 faster than light/backwards in time

Suppose there exists a particle that can travel faster than light (Tachyon)



0p is a faster than light path

In frame S, op is forwards in time $t_p > t_0$

In frame S', 0P is backwards in time, $t'_p < t'_0$

$\Rightarrow$ If a faster than light signal is possible, Then in some frames it is forwards in time, but in other frames it is backwards in time

$\tan\theta = \frac{v}{c}$

$\tan\alpha = \frac{c}{u}$

Where u is the speed of the tachyon

Backwards in time in S' if the relativity velocity of the frames satisfies

$$\tan\theta > \tan\alpha$$

$$\Rightarrow \frac{v}{c} > \frac{c}{u} \Rightarrow v > \frac{c^2}{u}$$

Directly from Lorentz transformation

For 0p,

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

$$x' = \gamma\left(x - \frac{v}{c}ct\right)$$

Speed of tachyon $u = \frac{x}{t}$

$t' < 0 \; if \; ct - \frac{v}{c}x < 0$

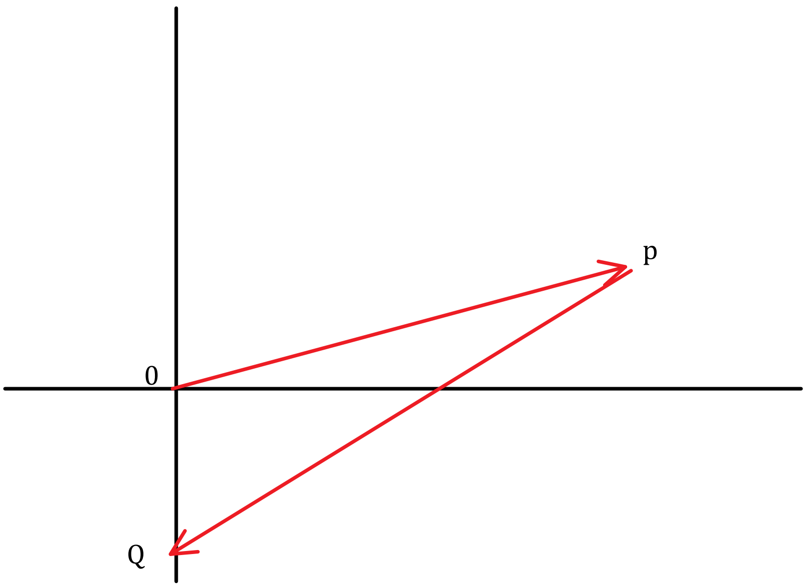
$$\Rightarrow ct\left(1 - \frac{v}{c^2}\frac{x}{t}\right) < 0$$

$$\left(1 - \frac{v}{c^2}\frac{x}{t}\right) \Rightarrow 1 - \frac{v}{c^2}u$$

$$\Rightarrow v > \frac{c^2}{u}$$

So far, this is not a terminal problem

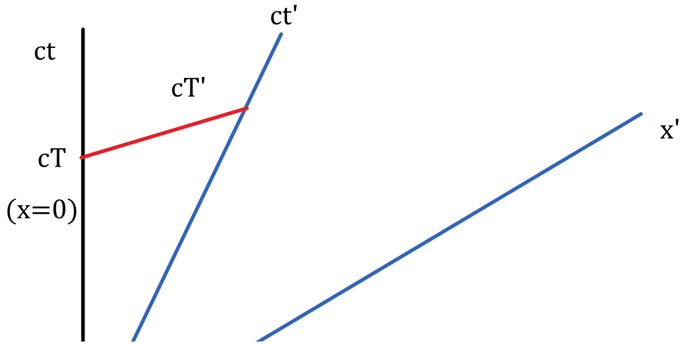
The problem is that once we accept faster then light/backwards in time in some frame is possible, Postulate I says that it is possible in all inertial frames in particular, it implies that if 0p is possible, then so is motion PQ

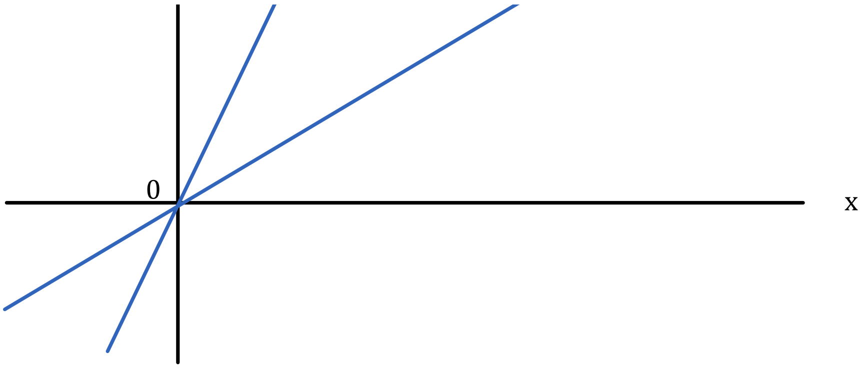


Closed path 0PQ is incompatible with causality ("grandfather" paradox)

3.3 Time dilation

Measure time differences in frames S and S'





In frame S, time interval is T
In frame S', time interval is T'
Where

$$T' = \gamma\left(T - \frac{vx}{c^2}\right) \Rightarrow \boxed{T' = \gamma T}$$

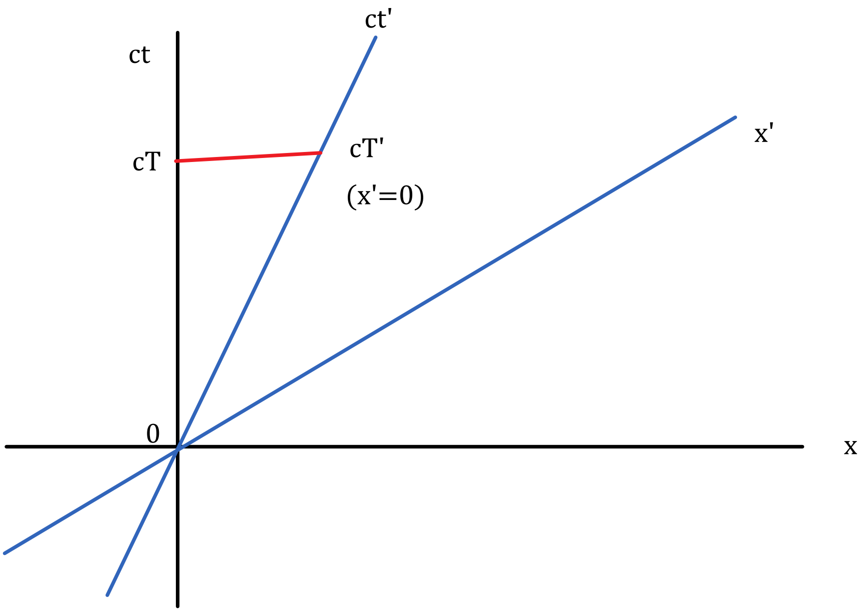
$$(\text{since } \frac{vx}{c^2} = 0)$$

Remember

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} > 1$$

$$\Rightarrow T' > T$$

This is Time Dilation
Now, consider instead a time difference between the events in the same place in S'



Frame S', time interval =T'
Frame S, time interval =T
Where

$$T = \gamma\left(T' + \frac{v}{c^2}x\right)$$

$$\Rightarrow T = \gamma T'$$

So $T > T'$
This must happen because postulate I says all inertial frames are the same
⇒ relation between measurements in S and S' must be symmetric
NB: this means that the minimum time between 2 events is the time measured in the frame where the events are at the same position

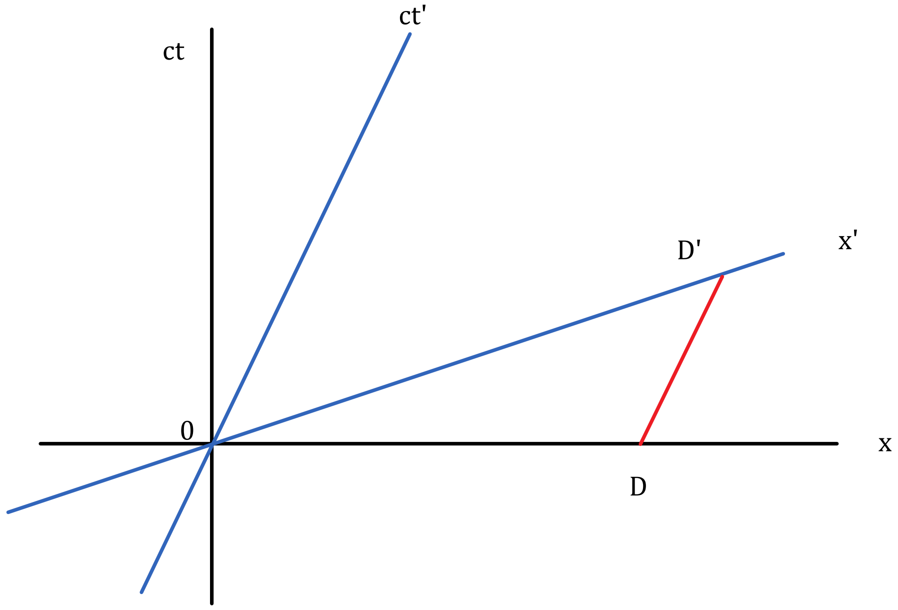
3.4 Length Contraction

Be very careful to be precise about what is being measured

1. Space dilation
   This is similar to time dilation

ct'

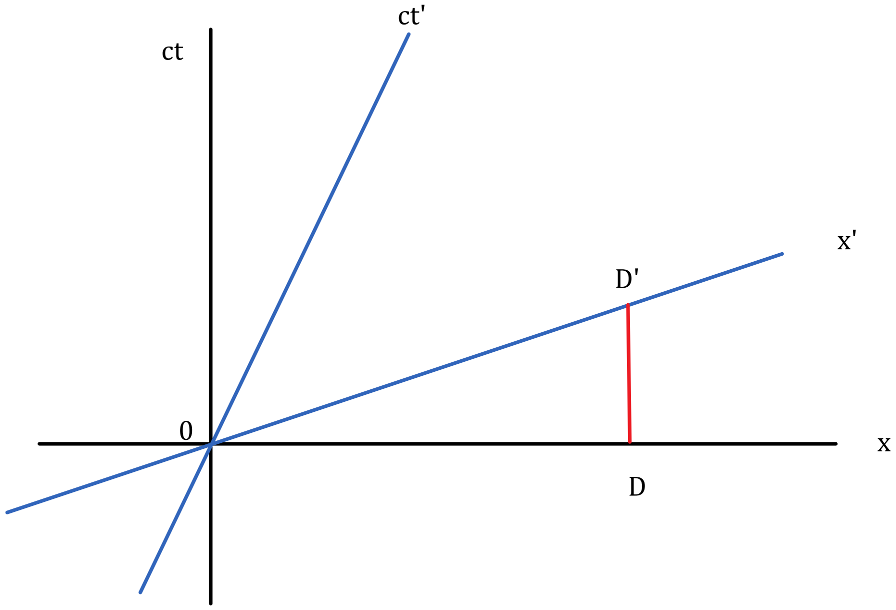
1. <u>Space dilation</u>
   This is similar to time dilation



Frame S, distance= D
Frame S', distance = D'
Where

$$D' = \gamma\left(D - \frac{vt}{c^2}\right) = \gamma D$$

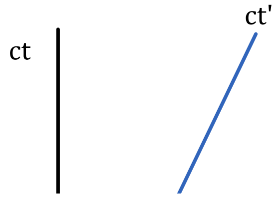
Similarly

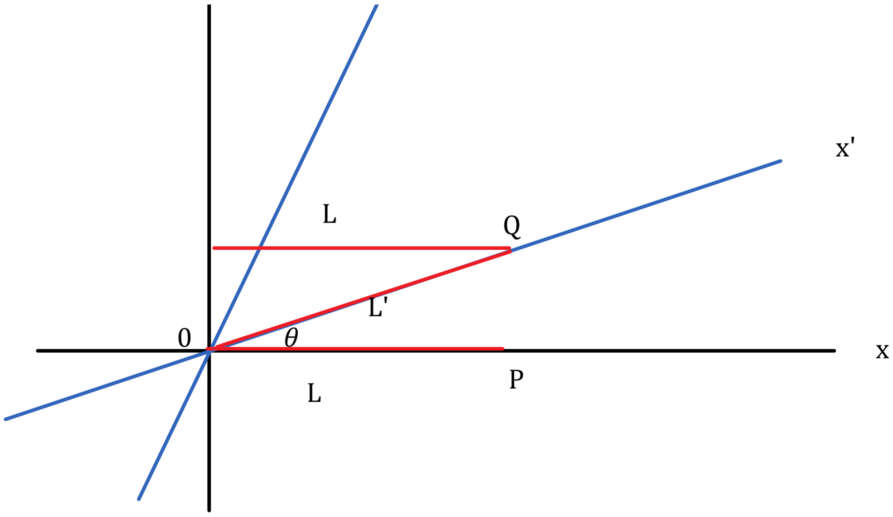


Frame S', Distance =D'
(measured at same t')
Frame S, distance=D

$$D = \gamma\left(D' + \frac{vt'}{c^2}\right) = \gamma D' > D'$$

Length Contraction
We need a definition of a measurement of length
Consider a rod f length L in frame S
Defn of length:
Length=distance between ends of rod measured AT THE SAME TIME





$\Rightarrow Length\ S, \quad L = x_P - x_0$

But in S', $L' = x'_Q - x'_0$

So in frame S,

$x_q = L$

$ct_Q = L \tan\theta = L\frac{v}{c}$

$\Rightarrow$ in frame S'

$x'_Q = \gamma(x_Q - vt_Q)$

$t'_Q = \gamma\left(t_Q - v\frac{x_Q}{c^2}\right) = 0$

$\Rightarrow L' = \gamma\left(L - \frac{v^2}{c^2}L\right)$

Since

$t'_Q = 0 \Rightarrow \frac{v}{c^2}x_Q$

$= \gamma\left(1 - \frac{v^2}{c^2}\right)L$

$\Rightarrow \boxed{L' = \gamma^{-1}L}$

Since $\gamma > 1, L' < L$

This is length contraction

Pole and garage "paradox"

Man carrying pole of length L' runs (fast) into a garage of length G garage is shorter than the pole, G<L'

Paradox??

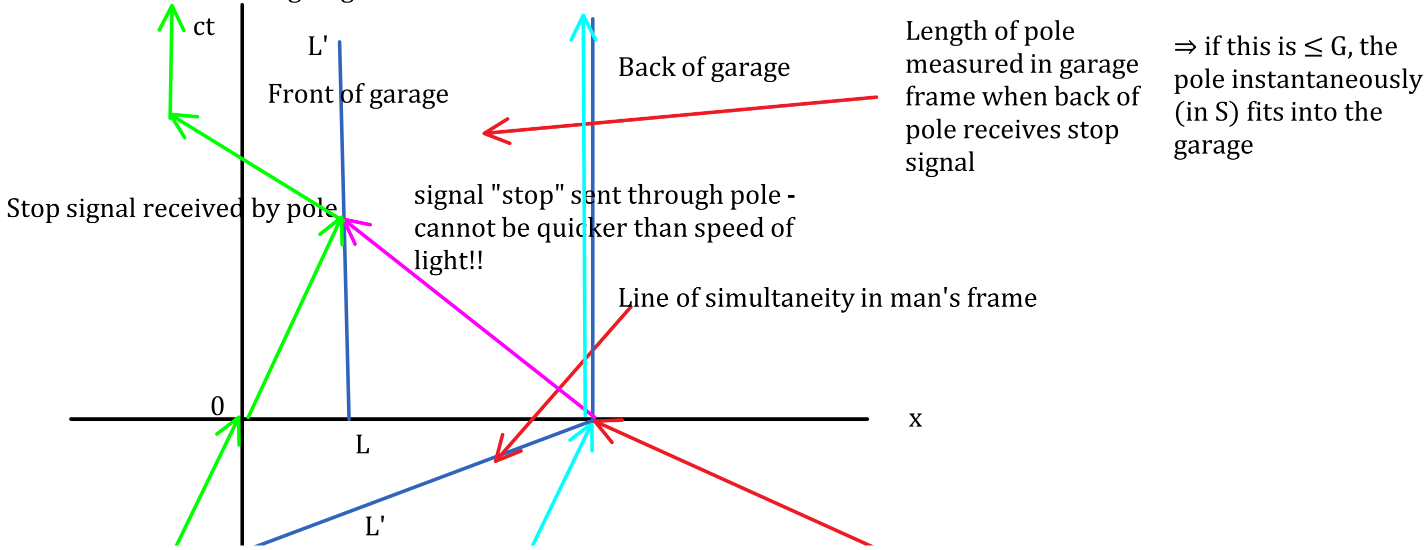
Garage frame- pole length is $L = \gamma^{-1}L' < L'$

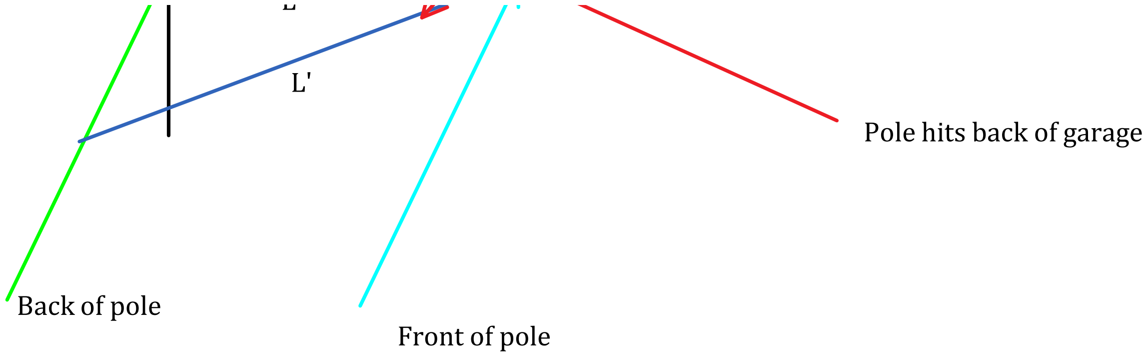
Can have L<G

Pole fits inside the garage

Man's frame- garage is length contracted

⇒Pole does not fit in garage





Key points
Because signal speed is at most C, the back of the pole doesn't receive stop signal instantaneously, so keeps moving ⇒ pole is being physically compressed!
It is because the pole is being compressed that it can (for a moment) fit into the garage
Then it expands back to its normal length, and comes to rest outside the garage
NB

"rigid bodies" are incompatible with special relativity because signal speeds are always <c

What is the max length of pole that fits into garage?

$G = ct$

$L - G = vt$

$\Rightarrow L = \left(1 + \frac{v}{c}\right) G$

But

$L = \gamma^{-1} L'$

$\Rightarrow L' = \gamma \left(1 + \frac{v}{c_G}\right)$

↑ longest pole that just fits into garage at speed v

<u>3.5 Velocity Addition Rule</u>

Suppose a particle moves at velocity u in frame S
What is it's velocity u' in frame S'
S' has velocity v relative to S
In 1 dim
Lorentz transf

$x' = \gamma(x - vt)$

$t' = \gamma\left(t - \frac{v}{c^2} x\right)$

So for constant speed

$$u' = \frac{x'}{t'} = \frac{\gamma(x - vt)}{\gamma\left(t - \frac{v}{c^2} x\right)} = \frac{\gamma\left(\frac{x}{t} - v\right)}{\gamma\left(1 - \frac{v}{c^2}\frac{x}{t}\right)}$$

$$\Rightarrow u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

This replaces the newtonian result

$u' = u - v$

This is a good approximation in the limit where u,v are much less than c

In 3dim
Let S' have velocity v along the x acis relative to S
Lorentz transformation

$x' = \gamma(x - vt)$

$y' = y$

$z' = z$

$t' = \gamma\left(t - \frac{v}{c^2} x\right)$

$$\Rightarrow u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c}}$$

$$u_y' = \frac{y'}{t'} = \frac{y}{\gamma\left(t - \frac{vx}{c^2}\right)} = \frac{uy}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

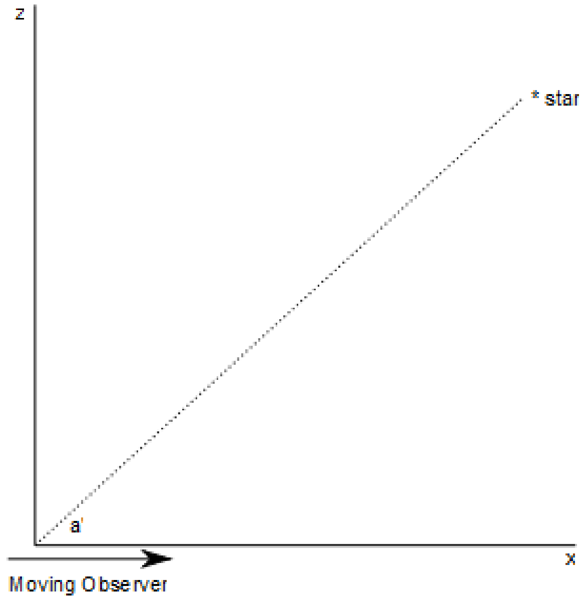
$$u_z' = \frac{uz}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

1. This can be turned into the velocity addition formula
   Particle 1 moves with velocity u_1 relative to the laboratory, and particle 2 moves with velocity u_2 relative to particle 1
   ⇒particle 2 speed in lab frame
   $$\boxed{= \frac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}}}$$

   Conclude if u_1 *and* u_2 are both less than c, so is
   $$\frac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}}$$
2. Consistency with postulate IB
   Speed of light is the same in all frames
   $$\frac{u_1 - u_2}{1 - \frac{u_1 u_2}{c^2}}$$
   If
   $$u = c \Rightarrow u' = \frac{(c - v)}{1 - \frac{cv}{c^2}} = c$$
3. From
   $$\frac{u - v}{1 - \frac{uv}{c^2}}$$
   Same simple algebra shows that
   $$\gamma(u') = \gamma(u)\gamma(v)\left(1 - \frac{uv}{c^2}\right)$$

Stellar aberration
Apparent position of stars traces a small ellipse over the course of the earth's orbit around the sun
This effect can be calculated accurately using the velocity addition formula

Suppose observer moves towards the star with velocity v
In moving frame
$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = -c\cos\alpha'$$
$$u_z' = \frac{u_z}{1 - \frac{u_z v}{c^2}} = -c\sin\alpha'$$



$$\Rightarrow \cos\alpha' = \frac{\cos\alpha + \frac{v}{c}}{1 + \frac{v}{c}\cos\alpha}$$

$$\sin\alpha' = \frac{\sin\alpha}{\gamma\left(1 + \frac{v}{c}\sin\alpha\right)}$$

We can tidy this up with a trig identity

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha}$$

3.6 Doppler Effect

Frequency of light detected from a moving object is increased/decreased if the object is moving towards/away from the observer
This is the Doppler effect

In Newtonian dynamics, there are two different Doppler formulae depending on whether

a. Source moves, detector stationary
b. Detector moves, source stationary

In special relativity, only the relative motion of source and detector is real⇒ there is only one Doppler formula that depends on the relative velocity

Consider a source moving with velocity v away from an observer (eg distant galaxy receding from earth)
Let the time at source between successive maxima of wave be $dt_0 = \frac{1}{v_0}, where\ v_0$ =freq at source
The observer measures a time difference $dt + \frac{v}{c}dt$ → because source has moved
Where dt=time between wave maxima measured by observer
But special relativity (time dilation) ⇒ $dt = \gamma dt_0$
So

$$dt_{obs} = dt + \frac{v}{c}dt = \gamma\left(1 + \frac{v}{c}\right)dt_0$$

$$\frac{v_{obs}}{v_0} = \frac{dt_0}{dt_{obs}} = \gamma^{-1}\left(1 + \frac{v}{c}\right)^{-1} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v}{c}}$$

time dilation gives extra factor γ

$$\Rightarrow \boxed{\frac{v_{obs}}{v_0} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}}$$

Relativistic Doppler formula
V +'ve (source moving away) ⇒ $v_{obs} < v_0$
redshift

V -'ve (source approaching)$\Rightarrow v_0 < v_{obs}$

Blueshift

Transverse Doppler effect

In special relativity (only) there is a doppler effect even when the source is moving orthogonal to the direction of the signal

$$\left.\frac{v_{obs}}{v_0}\right|_{transverse} = \gamma^{-1}$$

4.1 Atomic Clocks and Time Dilation

07 November 2011
15:42

1971 Hafele & Keating fly atomic clocks around the world in opposite directions & compare
SR⇒ clocks following different paths so should register different times,
Clock going east: 59ns slow
West: 273 ns fast

<u>4.2 Muon decay</u>
Lifetime of a muon at rest is $\sim 10^{-6}$ s
$\mu^- \rightarrow e^- \bar{v}_e v_\mu$

In 1966, CERN (small storage ring ~ 7m radius) with speeds 0.997c ($\Leftrightarrow \gamma = 12$)
Lifetime was increased by a factor of 12
Time dilation
Repeated in 1978 with $\gamma = 29$
For comparison, at LEP with energies 50GeV/beam γ factor is 10^5

5 Relativistic dynamics

14 November 2011
10:08

5.1 Vectors in Minkowski Spacetime

We formulate dynamics in special relativity using the language of 4-vectors
Prototype 4-vector is the position 4-vector,

$$x^\mu = (ct, \underline{x})$$
$$= (ct, x, y, z)$$
$$\mu = \quad 0, \quad 1, \quad 2, \quad 3$$
$$\quad time \quad space \quad space \quad space$$

Because it is a 4-vector, x^μ transforms as

$$x'^\mu = L^\mu_v x^v$$

$$L^\mu_v = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & & \\ -\gamma\frac{v}{c} & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$
$$x' = \gamma\left(x - \frac{v}{c}ct\right)$$
$$y' = y$$
$$z' = z$$

ANY 4-vector has the same lorentz transformations
We can make a quantity out of x^μ which is Lorentz invariant (doesn't change under a Lorentz transformation)

$$s^2 = g_{\mu v} x^\mu x' = -c^2t^2 + x^2 + y^2 + z^2 = -c^2t^2 + \underline{x} * \underline{x}$$

$$g_{\mu v} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

5.2 4-velocity

If we try to define the 4-velocity as

$$U^\mu \overset{?}{=} \frac{dx^\mu}{dt}$$

But this does NOT have the proper Lorentz transformation to be a 4-vector
To make a 4-vector we need to differentiate with respect to something which is itself Lorentz invariant.

$$S^2 = -c^2t^2 + \underline{x} * \underline{x}$$

IS Lorentz invariant

Define the proper time τ, where $S^2 = -c^2\tau^2$

τ is the actual time measured in a co-moving frame i.e. along the particles path
So define the velocity 4-vector as

$$U^\mu = \frac{dx^\mu}{d\tau}$$

So U^μ has the correct Lorentz transformations
Components-

$$U^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt}$$

Now

$$-c^2\tau^2 = -c^2t^2 + x^2 + y^2 + z^2 = -c^2t^2\left(1 - \frac{x^2}{c^2t^2} - \frac{y^2}{c^2t^2} - \frac{z^2}{c^2t^2}\right)$$

$$-c^2t^2\left(1 - \frac{U_x^2}{c^2} - \frac{U_y^2}{c^2} - \frac{U_z^2}{c^2}\right)$$

$= -c^2t^2\left(1-\frac{u^2}{c^2}\right)$
$= -c^2t^2\gamma^{-2}(U)$
$\Rightarrow -c^2\tau^2 = -c^2t^2\gamma^{-2}(u)$
$\Rightarrow \tau = t\gamma^{-1}(u)$
$\Rightarrow \boxed{\frac{dt}{d\tau} = \gamma(u)}$
This implies that
$U^\mu = \frac{dx^\mu}{d\tau} = \gamma(u)\frac{dx^\mu}{dt}$
$\Rightarrow \boxed{U^\mu = (\gamma(u)x, \gamma(u)\underline{u})}, since\ x^\mu = (ct, \underline{x})$
Check Invariant
$g_{\mu v}U^\mu U^v = -U^0U^0 + U^1U^1 + U^2U^2 + U^3U^3 = \gamma^2(-c^2 + u^2) = -c^2$

5.3 4-momentum and mass

The 4-momentum is defined as
$P^\mu = mU^\mu$
So the mass M is a Lorentz invariant quantity (!!)
P^μ has the correct Lorentz transformations to be a 4-vector
Its components are
$\boxed{P^\mu = (\gamma(u)mc, \gamma(u)m\underline{u})}$
The invariant formed from P^μ is
$g_{\mu v}P^\mu P^v = -\gamma^2m^2c^2 + \gamma^2m^2\underline{u} * \underline{u} = -m^2c^2$
So
$P^2 \equiv g_{\mu v}P^\mu P^v = -m^2c^2$
That is, mass is the invariant quantity made from the 4-momentum (just like spacetime interval S^2 made from the position 4-vector x^μ
Interpret components of the 4-momentum
Write

$P^\mu = (P^0, \underline{p})$

$\underline{p} = 3 - momentum$

$\Rightarrow \boxed{\underline{p} = \gamma(u)m\underline{u}}$
NB This is different from the usual Newtonian definition by the factor $\gamma(u)$
The "timelike" component
$P^0 = \gamma(u)mc = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}mv$
For small velocities u<<c, expand
$\gamma(u) = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{u^2}{c^2} + O\left(\frac{u^\mu}{c^\mu}\right)$

o=order?

$\Rightarrow P^2 = 1 + \frac{1}{2}\frac{u^2}{c} + \cdots$

$\leftrightarrow cP^2 = mc^2 + \frac{1}{2}mu^2$
Notice that the newtonian kinetic energy is 1/2 m
This motivates use to identify cP^0 as energy E
Thus

$P^\mu = \left(\frac{E}{c}, \underline{P}\right)$
$E = \gamma(u)mc^2$
$\underline{P} = \gamma(u)m\underline{u}$

4-momentum

$P^\mu = \left(\frac{E}{c}, \underline{p}\right)$

Where

$\underline{p} = \gamma(u)m\underline{u}$
$E = \gamma(u)mc^2$

NB

(1) 3-momentum $\underline{p}$ in special relativity is NOT $\underline{p} = m\underline{u}$ but $\boxed{\underline{p} = \gamma(u)m\underline{u}}$

For small velocities only $(u \ll c)$, $\underline{p} \approx m\underline{u}$

(2) Identify $\gamma(u)mc^2$ as energy E

For small $u \ll c$

$$\gamma(u) = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

$$\Rightarrow E = mc^2 + \frac{1}{2}mu^2 + \cdots$$

$$\frac{1}{2}mu^2 \rightarrow newtonian\ kinetic\ energy$$

$$mc^2 \rightarrow new\ in\ special\ relativity$$

$E \neq 0$ even when particle is at rest

Call this the "rest energy",

$$E_{rest} = mc^2$$

If the particle is moving

$$E = \gamma(u)mc^2$$

$E = mc^2$ opens the possibility of extracting energy from reactions where the total mass of constituents change

<u>Lorentz transformations</u>

We already know the lorentz transformations for any 4-vector. They are exactly the same as for the position 4-vector

$x^\mu = (ct, \underline{x})$

Dictionary

$x^\mu \leftrightarrow p^\mu$

$ct \leftrightarrow \frac{E}{c}$

$\underline{x} \leftrightarrow \underline{p}$

$g_{\mu\nu}u^\mu u^\nu \leftrightarrow g_{\mu\nu}p^\mu p^\nu$

So we can immediately write the Lorentz transformations for energy and 3-momentum

$P'^\mu = L^\mu_\nu p^\nu$

$$\begin{cases} E' = \gamma(v)(E - vp_x) \\ P'_x = \gamma(v)\left(p_x - \frac{v}{c^2}E\right) \\ p'_y = p_y \\ p'_z = p_z \end{cases}$$

For a transformation between frames S and S' with relative velocity v in the x-direction

This shows that BOTH energy and momentum change when measured in different frames

As with any 4-vector, we can construct a Lorentz invariant quantity from $p^\mu = \left(\frac{E}{c}, \underline{p}\right)$

Lorentz invariant is

$$g_{\mu\nu}p^\mu p^\nu$$
$$= -p^0p^0 + p^1p^1 + p^2p^2 + p^3p^3$$
$$= -\frac{E^2}{c^2} + p_xp_x + p_yp_y + p_zp_z$$
$$= -\frac{E^2}{c^2} + \underline{p} * \underline{p}$$

So the quantity

$$\left(-\frac{E^2}{c^2} + \underline{p} * \underline{p}\right)$$

Is invariant under Lorentz transformations

Evaluate

$$-\frac{E^2}{c^2} + \underline{p} * \underline{p} = -\gamma(u)^2 m^2 c^2 + \gamma(u)^2 m^2 u^2$$
$$= -\gamma^2(u)\left(1 - \frac{u^2}{c^2}\right) mc^2 = -m^2 c^2$$
→ so the Lorentz invariant quantity is the mass

We find
$$E^2 - c^2 \underline{p} * \underline{p} = m^2 c^4$$
$$\boxed{E^2 - c^2 p^2 = m^2 c^4}$$
This holds for any frame
CHECK explicitly
$E'^2 - c^2 p'^2 = m^2 c^4 m$

Using Lorentz transformations

<u>Photons</u>
Photons have zero mass
$\Rightarrow E^2 - c^2 p^2 = 0$
$$E = c|\underline{p}|$$
Photon 4-momentu
$$p^\mu = \left(\frac{E}{c}, \underline{p}\right)$$
For a massive particle, $p^\mu = (\gamma(u)mc, \gamma(u)m\underline{u})$
The formula for a photon is the singular limit
$m \to 0, \gamma(u) \to \infty$
$\Rightarrow u \to c$
This is a consequence of the fact that a massless particle (photon) <u>must</u> travel at the speed of light, $u = c$
<u>5.4 Postulates of relativistic dynamics</u>
To complete the formulation of dynamics in special relativity, we give the analogues of newton's laws
1a. Equivalence of all inertial frames (there aren't? global inertial frames)
1b. Speed of light is the same in all inertial frames
Introduces a new fundamental constant c into physics
2. This introduces the idea of a force 4-vector
$$F^\mu = \frac{dP^\mu}{d\tau}$$
$\tau =$ proper time
Generalises newtonian
$$\underline{f} = \frac{d\underline{P}}{dt}$$
$$\Leftrightarrow F^\mu = \gamma(u)\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\underline{P}}{dt}\right) = \left(F^0, \gamma(u)\underline{f}\right)$$
So in particular
$$\underline{f} = \frac{d\underline{p}}{dt}$$
In S. Rel
3. Dynamics takes place in spacetime.
Laws of physics are invariant under translations in space and time
Noether's theorem ⇒ conservation of 3-momentum and energy
i.e. invariance under translations in spacetime ⇒ conservation of 4-momentum p^μ

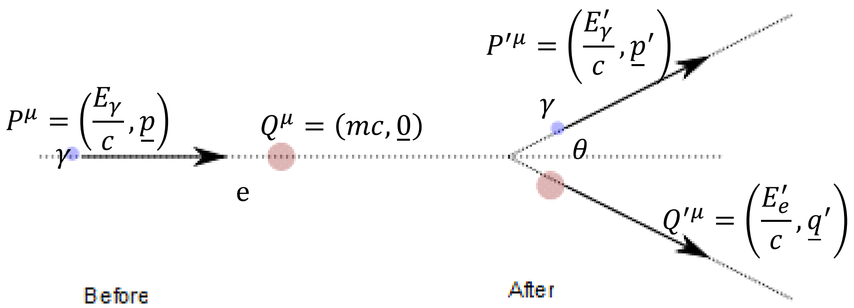
In practice, to solve problems in relativistic dynamics, we use two main tools
1) Energy +3 momentum conservation
2) Energy-momentum mass relation $E^2 - c^2|\underline{p}|^2 = m^2 c^4$
Remembering mass m is Lorentz invariant

6 Relativistic collisions

21 November 2011
10:25

6.1 Compton effect

Collision between a photon and electron, in which the electron is initially at rest
After collision, photon loses energy ⇒ wavelength increases (red shifted)
1922, Compton, using x-rays



Let

$$P^\mu = \left(\frac{1}{c}E_\gamma, \underline{p}\right)$$
$$Q^\mu = (mc, \underline{0})$$
$$mc = \frac{1}{c}E_e$$
$$\underline{q} = \underline{0}$$

And

$$P'^\mu = \left(\frac{1}{c}E'_\gamma, \underline{p}'\right)$$
$$Q'^\mu = \left(\frac{1}{c}E'_e, \underline{q}'\right)$$

Use energy and momentum conservation separately
$E'_e = E_e + E_\gamma - E'_\gamma$
$q' = \underline{q} + \underline{p} - \underline{p}$
Energy-momentum-mass relation for photons
$E_\gamma = c|\underline{p}|$
$E'_\gamma = c|\underline{p}'|$

Energy-momentum-mass relation for electron
$E'^2_e - c^2|\underline{q}'|^2 = m^2c^4$
But
$E'^2_e - c^2|\underline{q}'|^2 = \left(mc^2 + c|\underline{p}| - c|\underline{p}'|\right)^2 - c^2\left(|\underline{p}|^2 + |\underline{p}'|^2 - 2|\underline{p}||\underline{p}'|\cos\theta\right)$
$= m^2c^4 + 2mc^3\left(|\underline{p}| - |\underline{p}'|\right) + c^2\left(|\underline{p}| - |\underline{p}'|\right)^2 - c^2|\underline{p}|^2 - c^2|\underline{p}'|^{2\,2}{}_{2c}|\underline{p}||\underline{p}'|\cos\theta$
$= m^2c^4 + 2mc^3\left(|\underline{p}| - |\underline{p}'|\right) - 2c^2|\underline{p}||\underline{p}'|(1 - \cos\theta)$
Conclude

$$\boxed{mc\left(|\underline{p}| - |\underline{p}'|\right) = |\underline{p}||\underline{p}'|(1 - \cos\theta)}$$

Or,

$$\boxed{E_\gamma - E'_\gamma = \frac{1}{mc^2} E_\gamma E'_\gamma (1 - \cos\theta)}$$

So energy of scattered photon E'_γ depends on the scattering angle θ
Bigger scattering angle ⟺bigger energy loss
In quantum mechanics

$$E_\gamma = hv = \frac{hc}{\lambda}$$

$$\Rightarrow \boxed{\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)}$$

$E_\gamma E'_\gamma (1 - \cos\theta) = mc\left(E_\gamma - E'_\gamma\right)$
$\theta \approx 0$ small angle scattering $E'_\gamma \approx E_\gamma$
$\hat{\theta}$ bigger ⇔ Bigger energy loss, $E'_\gamma \ll E_\gamma$
This is "normal" Compton scattering, i.e. where the photon loses energy
"Inverse" Compton scattering is where the photon is back-scattered from a high-energy electron and gains energy.
This is important

1) Lab, to get high-energy photon beam
2) Astrophysics, i.e. gamma-ray burst

<u>Compton scattering with 4-momentum notation</u>
Notation

$$P.Q = g_{\mu u} P^\mu Q^u = -P^0 Q^0 + P^1 Q^1 + P^2 Q^2 + P^3 Q^3$$

$$g_{\mu u} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$g_{\mu u} P^\mu Q^u = P^\mu g_{\mu u} Q^u = (1 * 4)(4 * 4)(4 * 1)$$
$$= 1 * 1 = number$$

$$P^2 = g_{\mu u} P^\mu P^u$$

So, e.g.

$$Q' = \left(\frac{E'_e}{c}, \underline{q}'\right)$$

$$Q'^2 = \frac{1}{c^2} E'^2_e + \underline{q}'.\underline{q}'$$
$$= -m^2 c^2$$

Similarly

$$P = \left(\frac{E_\gamma}{c}, \underline{p}\right)$$

$$P^2 = -\frac{E_\gamma^2}{c^2} + \underline{p}.\underline{p}$$

So, using energy-momentum conservation in Compton scattering
$P^\mu + Q^\mu = P'^\mu + Q'^\mu$
$\Rightarrow Q'^\mu = Q^\mu + P^\mu - P'^\mu$
So, using the energy-momentum-mass relation
$Q'^2 = Q^2 + 2Q.(P - P') + P^2 + P'^2 - 2P.P'$

$Q'^2 = -m^2 c^2$
$Q^2 = -m^2 c^2$
$P^2 = P'^2 = 0$

$Q.(P - P') = P.P'$
$-mc\left(\frac{1}{c} E_\gamma - \frac{1}{c} E'_\gamma\right) =$
Since

$$P.P' = -\frac{E_\gamma E'_\gamma}{c^2} + \underline{P}.\underline{P}' = -\frac{E_\gamma E'_\gamma}{c^2} + |\underline{P}||\underline{P}|\cos\theta$$
$$= -\frac{E_\gamma E'_\gamma}{c^2}(1-\cos\theta)$$
$E_\gamma E'_\gamma(1-\cos\theta) = mc^2(E_\gamma - E'_\gamma)$

Reproducing the results already found

NB equivalence between these steps and those in the original derivation

6.2 Colliding Beams

Modern particle fall into two categories- colliders and fixed target

In a collider, 2 beams are accelerated and stored in counter-rotating rings, then collided head-on

Examples

LHC	P P	3.5 TeV beams
LEP	e^+e^-	50-100GeV
$Sp\bar{p}S$	$p\bar{p}$	270GeV
		CERN
HERA	e^-p	
		DESY

+lots of other lower-energy e^+e^-

CERN

ISR,

Pp

Few GeV

In a fixed target accelerator, a high-energy beam of particles is scattered from a stationary target

CERN: SPS p beam→ □

270GeV beam

(super proton synchrotron)

Original PS ~27GeV

6.2 Colliding beams (cont)

The simplest example of a particle collider is LEP, which collided beams of e^+ and e^- with equal energies, initially $E_{beam} \approx 50GeV$

Two electrons

$P_2^\mu = \left(\frac{E}{c}, -\underline{p}\right)$

$P_1^\mu = \left(\frac{E}{c}, \underline{p}\right)$

$E = E_{beam}$

Energy-momentum-mass relation

Recall $P^2 = g_{\mu v}P^\mu P^v$

$$= -P^0P^0 + \underline{P}.\underline{P} = -\frac{E^2}{c^2} + |\underline{P}|^2$$

$P_1^2 = -\frac{E^2}{c^2} + |\underline{P}|^2 = -m^2c^2$

$P_2^2 = -\frac{E^2}{c^2} + |\underline{P}|^2 = -m^2c^2$

Total 4-momentum

$$P_T^\mu = P_1^\mu + P_2^\mu = \left(\frac{2E}{c}, \underline{0}\right)$$

This is in the lab frame. In this special case (only), this is also the Centre of Momentum (CM) frame

In general, the CM frame is defined as the frame of reference where the total 3-momentum is zero.

We can always find this frame by making an appropriate Lorentz transformation.
In the CM frame, the total energy E_T^{CM} is available to make new particles eg at LEP, $e^+e^- \to Z$
So at LEP

$$E_T^{CM} = E_T^{lab}$$
$$= 2 \times 50 = 100GeV$$

Maximum mass of new particle is

$$m - \frac{2E}{c^2}$$

Not all colliders are symmetric
Eg at DESY, the e^-p collider HERA collided beams of e^- with energy 26 GeV and with energy 840 GeV
Electron 4-momentum

$$P_1^\mu = \left(\frac{E_1}{c}, \underline{P_1}\right)$$

Proton 4-momentum

$$P_2^\mu = \left(\frac{E_2}{c}, \underline{P_2}\right)$$

Energy-momentum-mass relations,

$$P_1^2 = -\frac{E_1^2}{c^2} + |\underline{P_1}|^2 = -m_1^2c^2$$
$$m_1 = electron\ mass \approx 0.5MeV$$
$$P_2^2 = -\frac{E_2^2}{c^2} + |\underline{P_2}|^2 = -m_2^2c^2$$
$$m_2 = proton\ mass \approx 1GeV$$

Total 4-momentum in LAB frame

$$P_T^\mu = \left(\frac{E_1 + E_2}{c}, \underline{P_1} + \underline{P_2}\right)$$

By definition, in CM frame

$$P_{T\,CM}^\mu = \left(\frac{E_1 + E_2}{c}, \underline{0}\right) = \left(\frac{E_{CM}}{c}, \underline{0}\right)$$

Problem is to find E_{CM}
2 ways

1) Explicitly work out the velocity v of the CM frame S' relative to LAB frame S, so that $\underline{P_1'} + \underline{P_2'} = 0$ then calculate $E_1' + E_2'$, using Lorentz transformation. -usually relatively hard
2) The quantity $P_T^2 = -g_{\mu v}P_T^\mu P_T^v$ is Lorentz invariant, so is the same whether we evaluate in LAB frame S or the CM frame S'

LAB frame

$$P_T^2 = -\left(\frac{E_1 + E_2}{c}\right)^2 + (\underline{P_1} + \underline{P_2}) + (\underline{P_1} + \underline{P_2})$$
$$= -\frac{E_1^2}{c^2} - \frac{E_2^2}{c^2} - 2E_1E_2 + |\underline{P_1}|^2 + |\underline{P_2}|^2 + 2\underline{P_1}.\underline{P_2}$$
$$= -m_1^2c^2 - m_2^2c^2 - \frac{2E_1E_2}{c^2} - 2|\underline{P_1}||\underline{P_2}|$$

By definition, in CM frame

$$= \left(\frac{E_1 + E_2}{c}, \underline{0}\right) = \left(\frac{E_{CM}}{c}, \underline{0}\right)$$

CM frame

$$P_{T\,CM}^\mu = -\frac{E_{CM}^2}{c^2}$$

Since

$$P_{T\,CM}^2 = m_1^2c^4 + m_1^2c^4 + 2E_1E_2 + 2|\underline{P_1}||\underline{P_2}|c^2$$

In general, this is the final result. (remembering $|\underline{P_1}|$ and $|\underline{P_2}|$ are given in terms of energies E_1 and E_2 by $-\frac{E_1^2}{c^2} + |\underline{P_1}|^2 = -m_1^2c^2$ etc)
In practice, eg at HERA, masses are small compared to beam energies
Since

$m_1c^2 \ll E_1, m_2c^2 \ll E_2$, we can neglect m_1, m_2 and $c|\underline{P_1}| \approx E_1, c|\underline{P_2}| \approx E_2$

So to an excellent approximation

$$E_{CM}^2 \approx 4E_1E_2 \Rightarrow \boxed{E_{CM} \approx 2\sqrt{E_1E_2}}$$

This is the general result for an asymmetric, high-energy collider where $E_{beam} \gg mass$

Clearly, in the special case of equal energy-beams, $E_1 = E_2 = E_{beam}$

$\Rightarrow E_{CM} = 2E_{beam}$ as for LEP

<u>6.3 Fixed target accelerators</u>

Eg SPS (super-proton-synchrotron) at CERN

Proton beam $E_{beam} = 270GeV$

Collides with a fixed proton target

$$P_1^\mu = \left(\frac{E}{c}, \underline{P}\right)$$

Accelerated proton

$$P_2^\mu = (Mc, \underline{0})$$

Rest proton, m=mass of proton

$$\Rightarrow P_T^\mu = P_1^\mu + P_2^\mu = \left(\frac{E + Mc^2}{c}, \underline{P}\right)$$

In CM frame, by definition,

$$P_{T\,CM}^\mu = \left(\frac{E_{CM}}{c}, \underline{0}\right)$$

Lorentz invariant

$$P_{T\,CM}^2 = P_{T\,LAB}^2$$

Now,

$$P_{T\,CM}^2 = -\frac{E_{CM}^2}{c^2}$$

$$P_{T\,LAB}^2 = -\frac{(E + Mc^2)^2}{c^2} + |\underline{P}|^2$$

$$= -\frac{E^2}{c^2} + |\underline{P}|^2 - 2EM - M^2c^2$$

$$-\frac{E^2}{c^2} + |\underline{P}|^2 = -M^2c^2$$

$$= -2EM - 2M^2c^2$$

So we find

$$E_{CM}^2 = 2EMc^2 + 2M^2c^4$$

For high energy accelerators, $E_{beam} \gg Mc^2$

So an excellent approximation is

$$E_{CM}^2 \approx 2EMc^2$$

$$\Rightarrow \boxed{E_{CM} \approx \sqrt{2EMc^2}}$$

Compare collider

$$E_{CM} \approx 2\sqrt{E_1E_2}$$

So for a given beam energy, E_{CM} is much bigger for a collider

Energy conservation

$$E_1 + mc^2 = E_2 + E_3$$

3-momentum conservation

$$\underline{P_1} = \underline{P_3} + P_4$$

Where $\theta = \theta_3 + \theta_4$

$$\Rightarrow |\underline{P_1}|^2 = |\underline{P_3}|^2 + |\underline{P_4}|^2 + 2|\underline{P_3}||\underline{P_4}|\cos\theta$$

Consider a special case where the particle separate with equal energies

Assume $E_3 = E_4$

$\Rightarrow \underline{P_3} = \underline{P_4}$

$\Rightarrow \theta_3 = \theta_4$

By conservation of transverse momentum

So the conservation equations simplify

$E_1 + mc^2 = 2E_3$ (1)

$E_1^2 - m^2c^4 = 2(E_3^2 - m^2c^4)(1 + \cos\theta)$ (2)

Using $E_1^2 - c^2|\underline{p}_1|^2 = m^2c^4$ etc

Now solve these to find scattering angle θ as a function of the initial beam energy E_1

Algebra,

$1 \Rightarrow E_1^2 - m^2c^4 = (E_1 + mc^2)(E_1 - mc^2) = 4E_3(E_3 - mc^2)$

Compare 1 and 2

$\Rightarrow 4E_3(E_3 - mc^2) = 2(E_3^2 - m^2c^4)(1 + \cos\theta) = 2(E_3 + mc^2)(E_3 - mc^2)(1 + \cos\theta)$

$\Rightarrow 1 + \cos\theta = \dfrac{4E_3}{2(E_3 + mc^2)}$

$\Rightarrow \cos\theta = \dfrac{E_3 - mc^2}{E_3 + mc^2}$

Finally, substitute for E_3 using $2E_3 = E_1 + mc^2$

$\Rightarrow \boxed{\cos\theta = \dfrac{E_1 - mc^2}{E_1 + 3mc^2}}$

At low energies, $E_1 \approx mc^2$

$\Rightarrow \cos\theta \approx 0 \Leftrightarrow \theta \approx 90°$

So the particles scatter at right angles

this is the well known result in Newtonian dynamics (snooker without spin)

At high energies, $E_1 \gg mc^2$

$\Rightarrow \cos\theta \approx 1 \Leftrightarrow \theta \approx 0$

The higher the energy, the smaller the scattering angle. The particles are scattered into a narrow forward cone. This is a very general result in relativistic scattering.

<u>2+2 scattering in the CM frame</u>

The analysis above was for the lab frame. Now recover the same result in the CM frame

~~

Momenta are equal and opposite in x-direction

Suppose the relative velocity of the CM frame and lab frame is v

$\Rightarrow E_2^{CM} = \gamma(v)m$

Now we showed previously that the total CM energy for fixed-target scattering is

$E_{CM} = \sqrt{2m(E + m)}$

$E_1^{CM} = \dfrac{1}{2}\sqrt{2m(E + m)}$

And since $E_1^{CM} = \gamma(v)m$

$\Rightarrow \gamma(v) = \dfrac{1}{2m}\sqrt{2m(E + m)}$

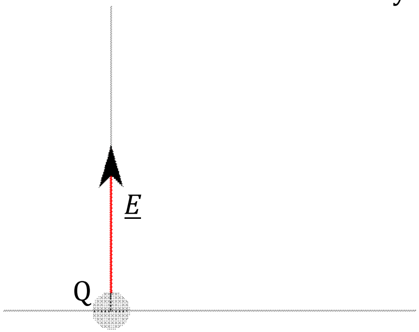
$\Rightarrow \sqrt{\dfrac{E + m}{2m}}$

This determines the velocity v at the CM frame relative to the LAB frame

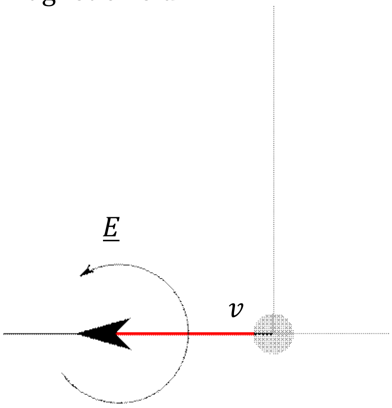
7 Electromagnetism

05 December 2011
10:20

A chae at rest in frame S has only an electric field.



But in frame S' (moving with velocity v in x-direction), this will appear to be a current ⇒ in S' there will also be a magnetic field



Electric & magnetic fields transform into each other as we change frames

Lorentz transformations
$E'_x = E_x : E'_y = \gamma(v)\left(E_y - vB_z\right) : E'_z = \gamma(v)\left(E_z + vB_y\right)$
$B'_x = B_x : B'_y = \gamma(v)\left(B_y + \frac{v}{c^2}E_z\right) : B'_z = \gamma(v)\left(B_z + \frac{v}{c^2}E_y\right)$

Identify E and B fields by there effect on a test charge q given by lorentz force $\underline{f} = q(\underline{E} + \underline{u} \times \underline{B})$
    $\underline{u}$ =Velocity of charge q
Since we know how force transforms between S and S', we can use the lorentz force law to deduce the transformation of E and B

7.1 Lorentz transformations for force
These are implicit in section 5 where we write the Lorentz transformations for the 4-force F^μ
Explicitly:-

$$\underline{f} = \frac{d}{dt}\underline{p} \Rightarrow f' = \frac{d}{dt'}\underline{p}' = \left(\frac{dt'}{dt}\right)^{-1}\frac{d}{dt}\underline{p}'$$

$p'_x = \gamma(v)\left(p_x - \frac{vE}{c^2}\right)$
$p'_y = p_y$
$p'_z = p_z$
$\Rightarrow \frac{d}{dt}p'_x = \gamma(v)\left(\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}\right)$
$\frac{d}{dt}p'_y = \frac{d}{dt}p_y$

Calculate $\frac{dE}{dt}$: −

$$E^2 = c^2 \underline{p}.\underline{p} + m^2 c^4$$
$$\Rightarrow 2E\frac{dE}{dt} = 2c^2\, \underline{p}.\frac{d\underline{p}}{dt}$$
$$\Rightarrow \frac{dE}{dt} = \frac{c^2 \underline{p}}{E}.\frac{d\underline{p}}{dt} - \underline{u}.\frac{d\underline{p}}{dt}$$

But

$$E = \gamma(u)mc^2$$
$$\underline{p} = \gamma(v)m\underline{u}$$

So

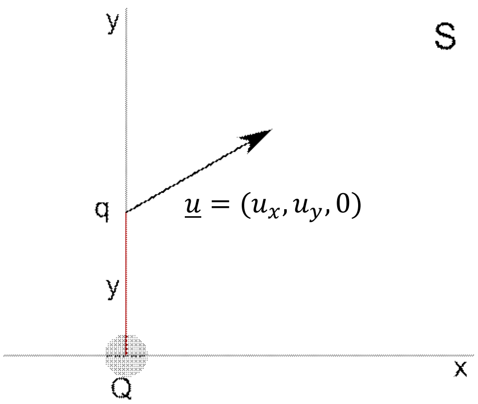
$$f_x' = \left(\frac{dt'}{dt}\right)^{-1} \gamma(v)\left(\frac{dp_x}{dt} - \frac{v}{c^2}\underline{u}\frac{d\underline{p}}{dt}\right)$$
$$= \frac{1}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)}\gamma(v)\left(f_x - \frac{v}{c^2}\underline{u}.\underline{f}\right)$$

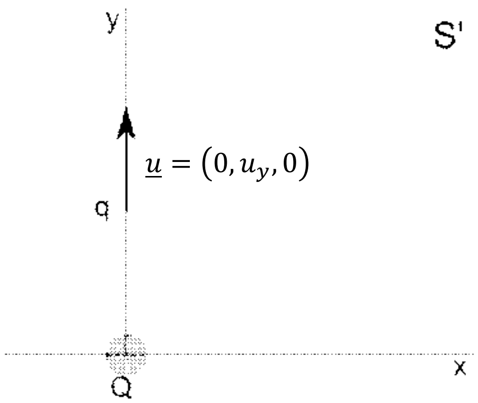
$$\boxed{f_x' = \frac{1}{\left(1 - \frac{u_x v}{c^2}\right)}\left(f_x - \frac{v}{c^2}\underline{u}.\underline{f}\right)}$$

$$\boxed{f_y' = \frac{1}{\left(1 - \frac{u_x v}{c^2}\right)} f_y, sim\ for\ f_z'}$$

<u>7.2 Electric and magnetic fields</u>

Consider a simple configuration of a charge Q with a test charge q





S' moving with velocity v w.r.t. S

In frame S, the force experienced by the test charge is

$$\underline{f} = (0, f_y, 0)$$

Where

$$f_y = q\epsilon, \epsilon = \frac{1}{4\pi\epsilon_0}\frac{Q}{y^2}$$

In frame S', the force is

$$f'_x = \frac{1}{\left(1 - \frac{u_x v}{c^2}\right)}\left(f_x - \frac{v}{c^2}\underline{u}.\underline{f}\right)$$

$$= \frac{1}{\left(1 - \frac{u_x v}{c^2}\right)}\left(f_x - \frac{v}{c^2}\underline{v}f_x - \frac{v}{c^2}u_y f_y\right) = -\gamma(v)^2\frac{vu_y}{c^2}q\epsilon$$

$$= -\gamma(v)\frac{vu'_y}{c^2}q\epsilon$$

$$f'_y = \frac{1}{\left(1 - \frac{u_x v}{c^2}\right)}f_y = \gamma(v)q\epsilon$$

$$f'_z = 0$$

Now use the Lorentz force law to deduce $\underline{E}'$ and $\underline{B}'$ fields

$$\Rightarrow \underline{E}' = (0, \gamma(v)\epsilon, 0), \underline{B}' = \left(0,0,-\gamma(v)\frac{v}{c^2}\epsilon\right)$$

In frame S'

Since $\underline{f}' = q(\underline{E}' + \underline{u}' \times \underline{B}')$

Compare

$$\underline{E} = (0, \epsilon, 0), \underline{B} = (0,0,0)$$

In frame S

Check that the special case agrees with the general lorentz transformations for E and B given above

7.3 Electromagnetic field tensor

We have justified the following lorentz transformations for components of the electric & magnetic field vectors

$$E'_x = E_x \quad E'_y = \gamma(v)(E_y - vB_z) \quad E'_z = \gamma(v)(E_z + vB_y)$$

$$B'_x - B_x \quad B'_y = \gamma(v)\left(B_y + \frac{v}{c^2}E_z\right) \quad B'_z = \gamma(v)\left(B_z - \frac{v}{c^2}E_y\right)$$

We would like to describe these in the same way as for 4-vector eg

$$x' = Lx \Leftrightarrow x'^\mu = L^\mu{}_v x^v$$

$$P' = LP \Leftrightarrow P'^\mu = L^\mu{}_v P^v$$

Where

$$L^\mu{}_v = \begin{pmatrix} \gamma & -\frac{\gamma \mathrm{v}}{c} & & \\ -\frac{\gamma v}{c} & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

To incorporate the 6 components of $\underline{E}$ and $\underline{B}$, put them into an antisymmetric 4*4 matrix $F^{\mu v}$

$$F^{\mu v} = \begin{pmatrix} & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & & -B_z & B_y \\ \frac{1}{c}E_y & B_z & & -B_x \\ \frac{1}{c}E_z & -B_y & B_x & \end{pmatrix}$$

Geometrically, this is called a <u>Tensor</u>

We can check that the lorentz transformations are equivalent to

$$F' = LFL^T$$

$$\Leftrightarrow F'^{\mu v} = L^\mu{}_\rho F^{\rho\sigma}(L^T)_\sigma{}^v$$

$$\boxed{F'^{\mu v} = L^\mu{}_\rho L^v{}_\sigma F^{\rho\sigma}}$$

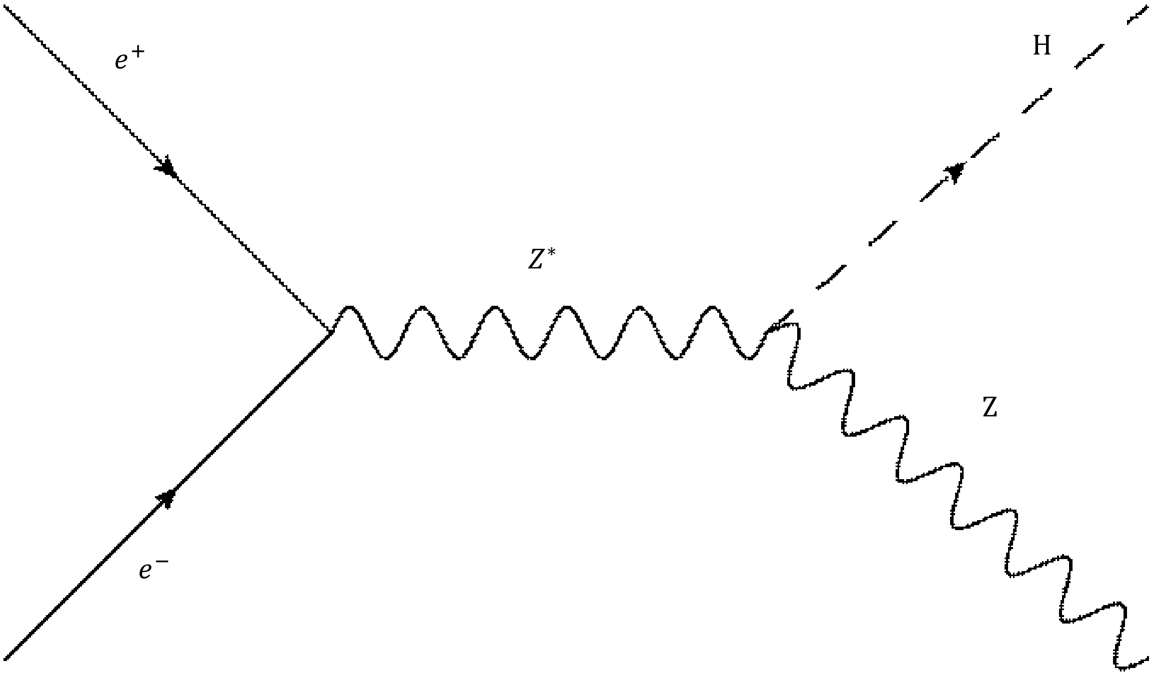
Note: Electromagnetism (Maxwell's equations) is already fully consistent with relativity, unlike Newtonian dynamics

No EM in jan exam

Higgs Boson

13 December 2011
10:28

LEP



$E_{CM} = 206GeV$
$E_{beam} = 103GeV$

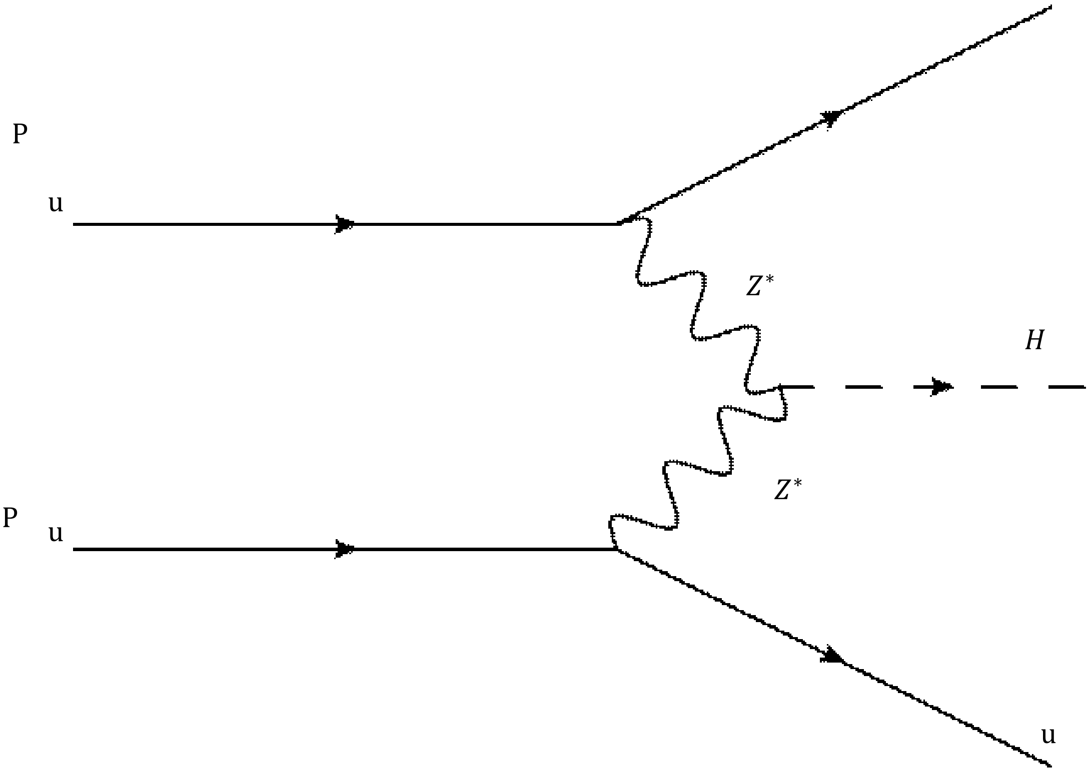
Limit
$m_H = 209 - 91$
$= 115GeV$

$m_z = 91GeV$

LEP didn't see H (shut down in 2000)
$\Rightarrow m_H > 115GeV$

LHC

u


P
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Z^*
H
Z^*
P
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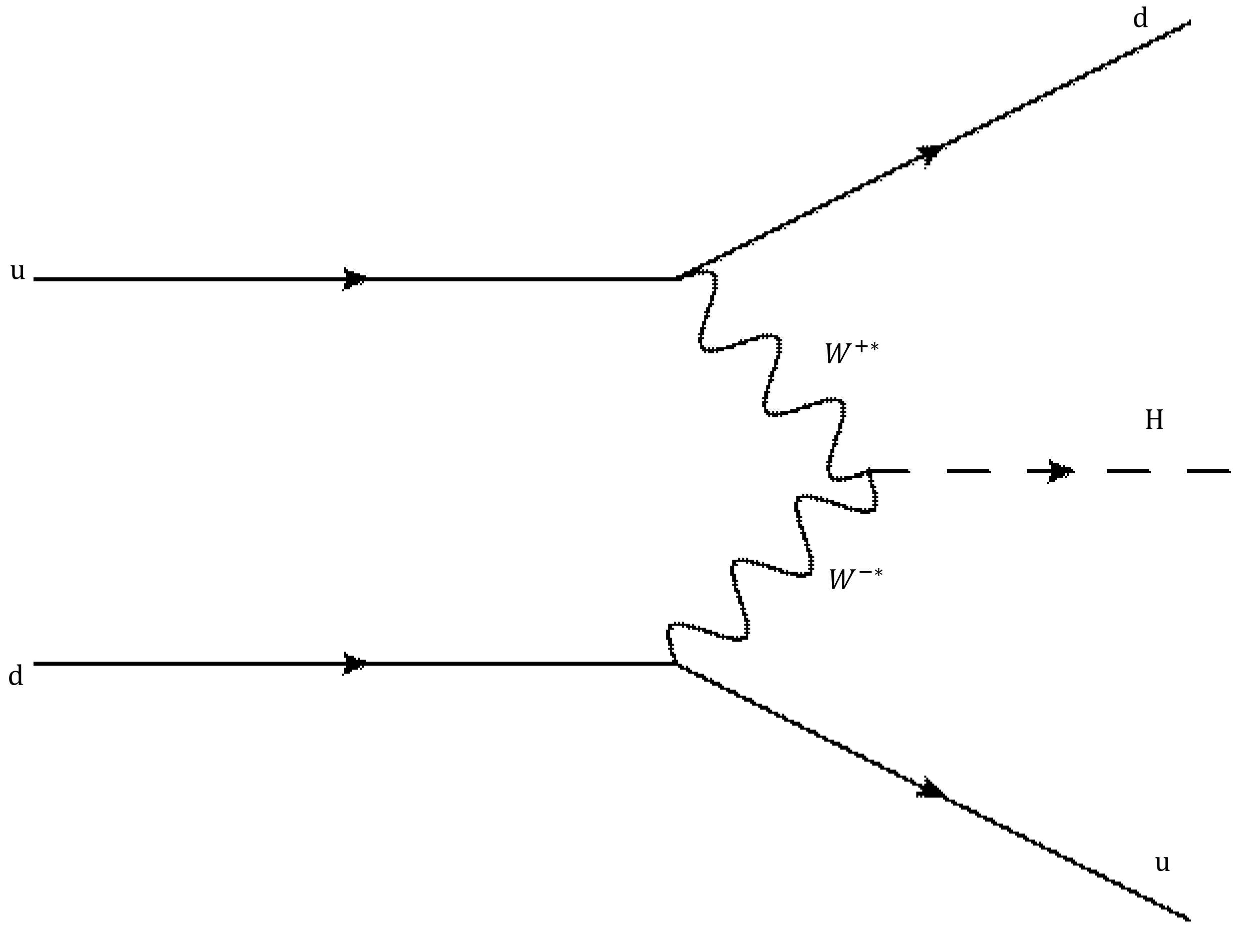
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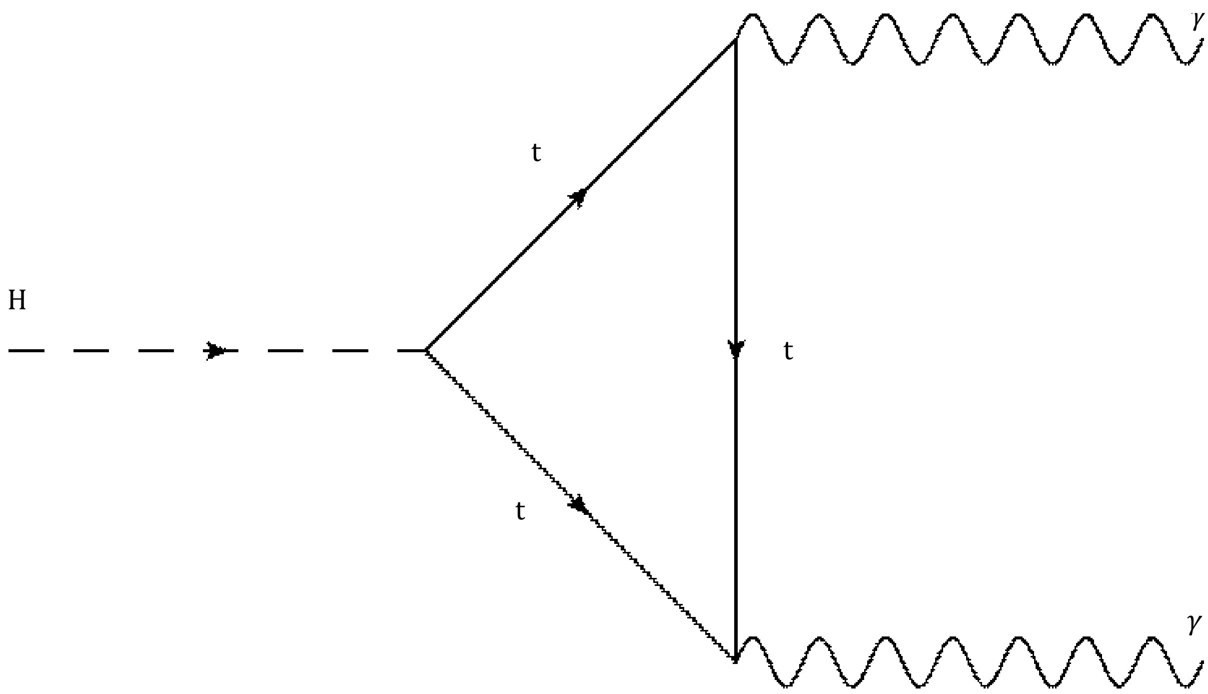

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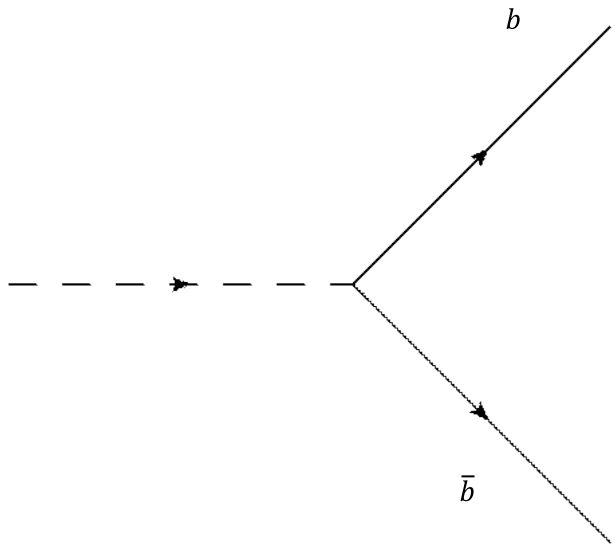


$\boxed{m_H \approx 126GeV?}$

Mathematical Methods

04 October 2011
10:55

Linear algebra
Vector calculus

Essential Mathematical Methods for the physical sciences
K.F. Riley and M. P. Hobson
CUP 2011
Cont assessment
4-5 exercise classes
~1 hour

MxN matrix: M rows and N columns

Vector $\bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Column vector

$\bar{w} = (w_1 \quad w_2 \quad w_3)$

Row vector

Transpose T
Swapping rows and columns
Trace of a matrix

$Tr\ A = \sum diagonal\ elements$

$$\sum_{i=1}^{N} a_{ii}$$

Multiplication
AB=C

$$c_{ij} = \sum_k a_{ik} b_{kj}$$

$AB \neq BA$

$Tr\ C = Tr\ AB\ =\ Tr\ BA$

$$Tr\ C = \sum_i c_{ii}$$

$$= \sum_i \sum_k a_{ik} b_{ki}$$

$$= \sum_{ijk} b_{ki} a_{ik}$$

$= tr\ BA$

$Tr\ BCA = tr\ CAB = tr\ BCA$

Linear Algebra
Vectorspace $\bar{v}$

$\bar{w} + \bar{v} = \bar{u}$ addition

$\lambda\bar{v} = \bar{w}$ multiplication

A+B=C provided that they are the same size

$\lambda A = B\ \ \lambda \in \mathbb{C}$

$(A + B)\bar{v} = A\bar{v} + B\bar{v}$

$A(\lambda\bar{v}) = \lambda A\bar{v}$

Products AB, $AA = A^2$

$\underbrace{A \quad \dots \quad A}_{n} = A^n$

Functions of matrices

e^A

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

⇒evaluate this bitches!

This is fundamental in QM

Heisenberg matrix mechanics

Hamiltonian H

Time evolution e^{iHt}

Book: section 1.1-1.5

<u>Inverse</u>

Numbers $y = ax$ $x = \frac{1}{a}y = a^{-1}y$

$\bar{y} = A\bar{x}$

$\bar{x} = A^{-1}\bar{y}$

2x2 example $y = Ax;\ \ x = A^{-1}y$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{cases} y_1 = ax_1 + bx_2 & .c \\ y_2 = cx_1 + dx_2 & .a \end{cases}$$

$$cy_1 - ay_2 = acx_1 + bcx_2 - acx_1 - adx_2 = (bc - ad)x_2$$

$$x_2 = \frac{1}{bc - ad}(cy_1 - ay_2)$$

$$x_1 = \frac{1}{bc - ad}(by_2 - dy_1)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A^{-1}y$$

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse exists except when

$ad - bc = 0$

A^{-1} exists, provided $\det A \neq 0$

Properties (general, verify for 2x2 matrices)

Product rule:

$\det AB = \det A \det B$

$A^{-1}A = I = identity\ matrix$ $(\det I = 1)$

$\det A^{-1}$

$\det A^{-1}A = \det I = 1$

$$= \det A^{-1} \det A \Rightarrow \det A^{-1} = \frac{1}{\det A}$$

$\det \kappa A$ multiply every matrix element by κ

$= \kappa^N \det A$

Minor:

M_{ij} =det(matrix with I'th row and j'th column deleted)

$$A = \begin{pmatrix} 2 & 3 & 1 \\ -4 & 8 & 0 \\ 10 & -1 & 5 \end{pmatrix}$$

$$M_{11} = \begin{vmatrix} 8 & 0 \\ -1 & 5 \end{vmatrix} = 40 - 0 = 40$$

$M_{12} = \begin{vmatrix} -4 & 0 \\ 10 & 5 \end{vmatrix} = -20$

$M_{13} = \begin{vmatrix} 2 & 1 \\ -4 & 0 \end{vmatrix} = 0 - -4 = 4$

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

$(-1)^{i+j} = 1\ if\ i + j\ is\ even, -1\ if\ odd$

$$\det A = \sum_{\substack{along\ row \\ or\ column}} A_{ij} C_{ij}$$

Ex: Expand along first row

$\det A = 2C_{11} + 3C_{12} + 1C_{13}$

$= 2(-1)^2 M_{11} + 3(-1)^2 M_{12} + 1(-1)^4 M_{13}$

$= 2 \times 40 \pm 3(-20) + 1(-76)$

$= 80 + 60 - 76 = 64$

Expand along second row

$\det A = -4C_{21} + 8C_{22} + 0C_{23}$

$= -4(-1)^3 M_{21} + 8(-1)^4 M_{22}$

$= 4 \begin{vmatrix} 3 & 1 \\ -1 & 5 \end{vmatrix} + 8 \begin{vmatrix} 2 & 1 \\ 10 & 5 \end{vmatrix}$

$= 4(15 + 1) + 8(10 - 10) = 64$

4x4 matrix

Minors are dets of 3x3 matrices

For 3x3

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{22}A_{13}A_{31}$

Special matrices

$$\begin{pmatrix} A_{11} & & & & \\ & A_{22} & & \emptyset & \\ & & A_{33} & & \\ & \emptyset & & \dots & \\ & & & & A_{NN} \end{pmatrix}$$

Diagonal matrix $diag\ (A_{11}, A_{22}, A_{33}, \dots, A_{NN})$

$\det A = A_{11}A_{22}A_{33} \dots A_{NN}$

$$A^{-1} = \begin{pmatrix} \frac{1}{A_{11}} & & & & \\ & \frac{1}{A_{22}} & & \emptyset & \\ & & \frac{1}{A_{33}} & & \\ & \emptyset & & \dots & \\ & & & & \frac{1}{A_{NN}} \end{pmatrix}$$

Upper|lower triangular

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ & A_{22} & A_{23} \\ & & A_{33} \end{pmatrix}$$

$\det(\square) = A_{11} \begin{vmatrix} A_{22} & A_{23} \\ & A_{33} \end{vmatrix} + 0 + 0 = A_{11}A_{22}A_{33}$

Transpose A^T

$(A^T)_{ij} = A_{ji}$

$\det(A^T) = \det A$

Complex conjugate *

A_{ij}^*

$\det(A^*) = (\det A)^*$

Hermitian conjugate

$A^{\text{\\dagger}} = A\ dagger = A^{T*}$

Symmetric matrices $A^T = A$
Antisymmetric: $A^T = -A$
Hermitian $A^{\text{\dagger}} = A$
Antihermitian $A^{\text{\dagger}} = -A$
Orthogonal matrices
$A^T A = 1$
Real matrix elements
2 vectors (real)
$\bar{x}, \bar{y}$
Interproduct
$\bar{y}^T \bar{x}$
$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ \bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$
$\bar{y}^T = (y_1 \quad y_2 \quad y_3)$
$\bar{y}^T \bar{x} = \bar{y} * \bar{x}$
$(y_1 \quad y_2 \quad y_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
$y_1 x_1 + y_2 x_2 + y_3 x_3$

$\bar{x} \to A\bar{x}$
$\bar{y} \to A\bar{y}$
$(AB)^T = B^T A^T$
$\bar{y}^T \bar{x} \to (A\bar{y})^T A\bar{x} = y^T A^T A\, \bar{x} = \bar{y}^T \bar{x}$
Orthogonal transformation preserves the interproduct rotations
$\det(A^T A) = \det 1 = 1$
$= \det A^T \det A = (\det A)$^2
$\det A = \pm 1\ (+1 \Rightarrow rotations$
Complex numbers
$\bar{x} \Rightarrow U\bar{x}\ unitary$
$\bar{y} \to U\bar{y}$
Unitary matrix
$U^{\text{\dagger}} U = 1$
Perserve inner product
$\bar{y}^T \bar{x} \to (Uy)^T Ux$
$= y^{dagger} u^{dagger} ux = y^{dagger} x$

Eigenvalues and Eigenvectors

11 October 2011
11:05

Ex

$$A = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

Consider a vector

$$\bar{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A\bar{w} = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Nothing special

Let's now take

$$\bar{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$A\bar{v} = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\bar{v}$ is an eigenvector of A

$A\bar{v} = \lambda\bar{v}$

A=matrix

λ= number (complex or real)

$\bar{v}$=eigenvector of A

λ is called an eigenvalue

⇒ appear all over physics

- Vibrations
- Crystals
- QM

Solve $A\bar{v} = \lambda\bar{v}\ \ find\ \lambda$

⇒ $(A - \lambda\mathbb{I})\bar{v} = 0$

Always $\bar{v} = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \end{pmatrix}$

Trivial solution: not interesting

Nontrivial solution

To have a nontrivial solution, should <u>not</u> be able to invert $(A - \lambda\mathbb{I})$

→ $\det(A - \lambda\mathbb{I}) = 0$

⇒polynomial eq in λ

⇒ $\boxed{characteristic\ equation}$

$N * N\ matrix: \lambda^N + \lambda^{N-1} + \cdots + \lambda + c = 0$

Coefficient of $\lambda^{N-1} = Tr\ A$

Check: proof will follow

$$Tr\ A = \sum_{i=1}^{N} \lambda_i\ (sum)$$

$$\det A = \prod_{i=1}^{N} \lambda_i\ (product)$$

In general, $\lambda \in \mathbb{C}$

$A\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

$= (a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + ad - bc = 0$

$$\lambda = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a+d)^2 - 4ad + 4bc}$$

$\lambda = \frac{(a+d)}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$
Complex if
$(a-d)^2 + 4bc < 0$
If b=c
$\lambda = \frac{(a+d)}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4b^2} > 0$
Real
Symmetric matrix has real eigenvalues
Ex
$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = A^T$$
Eigenvalues/eigenvectors
$A\bar{v} = \lambda v$\baar
1. λ eigenvalue
$$\det(A - \lambda \mathbb{I}) = 0$$
2. Find eigenvectors

Eigenvectors are determined uniquely except for the overall normalization. (they can have different magnitudes, but same direction)

If $\bar{v}$ is an eigenvector, then $\bar{v}' = \kappa\bar{v}$ is also eigenvector ($\kappa \in \mathbb{C}$)
$A\bar{v} = \lambda\bar{v}$
$A\bar{v}' = A(\kappa\bar{v}) = \kappa A\bar{v} = \kappa\lambda\bar{v} = \lambda(\kappa\bar{v}) = \lambda\bar{v}'$

Diagonal matrix
$A = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}, \lambda = 3: \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda = -3: \bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$
-eigenvalues in diagonal
We can transform A into D iva a transformation called diagonalization
Let's combine the eigenvectors in a matrix S
$$S = (\bar{v}_1 \quad \bar{v}_2)$$
$AS = A(\bar{v}_1 \quad \bar{v}_2)$
Statement:
Similarity transformation
$S^{-1}AS$
Diagonalizes A
$S = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$
$S^{-1} = \frac{1}{-3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$
Verify $S^{-1}S = \mathbb{I}$
$S^{-1}AS = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 3 & -6 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 9 & 0 \\ 0 & -9 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} = D$
$Tr\ AB = Tr\ BA$
$\Rightarrow tr(S^{-1}AS) = tr\ D = \sum_{i=1}^{N} \lambda_i$
$Tr(ASS^{-1}) = tr\ A$
$\det(AB) = \det A \det B$
$\det(S^{-1}AS) = \det D = \prod_{i=0}^{N} \lambda_i$
$\det S^{-1} \det A \det S = \det(SS^{-1}) \det A = \det A$
Ex
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$A^T = A$

$Tr\ A = 0$
$\det A = 0$
$\Pi\lambda = 0 \Rightarrow at\ least\ 1\ eigenvalue\ should\ be\ 0$
$\sum\lambda = 0 \Rightarrow \lambda = 0, +a, -a$

0
$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda = 0$$
Characteristic eq
$-\lambda(\lambda^2 - 2) = 0$
$$\lambda = \begin{matrix} 0 \\ \sqrt{2} \\ -\sqrt{2} \end{matrix}$$
Eigenvectors

1. $\lambda = 0$
$$A\bar{v} = 0\bar{v}, v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} y = 0 \\ x + 2 = 0 \\ y = 0 \end{matrix} \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$
2. $\lambda = \sqrt{2}$
$$A\bar{v} = \sqrt{2}\bar{v}$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$\begin{cases} y = \sqrt{2}x \\ x + z = \sqrt{2}y \\ y = \sqrt{2}z \end{cases} \qquad \begin{matrix} x + z = 2x \\ \Rightarrow x = z \\ y = \sqrt{2}x \end{matrix}$$
$$\bar{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$
3. $\lambda = -\sqrt{2}$
$$\bar{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\Rightarrow S, S^{-1} \quad S^{-1}AS = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$
Matrix A
Eigenvalues λ
$\det(A - \lambda\mathbb{I}) = 0$
Eigenvectors $\bar{v}$
$A\bar{v} = \lambda\bar{v}$
$$S = \begin{pmatrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \end{pmatrix}$$
$S^{-1}AS = D$
$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$
$$Tr\ A = \sum_i \lambda_i$$
$$\det A = \prod_i \lambda_i$$
$(AB)^{dag} = B^{dag}A^{dag}$
Hermetian matrix

$H^{dag} = H$
Suppose $\bar{v}$ is an eigenvector
$H\bar{v} = \lambda\bar{v}$
$(H\bar{v})^{dag} = (\lambda\bar{v})^{dag}$
$\bar{v}^{dag}H^{dag} = \lambda^*\bar{v}^{dag}$
$= \bar{v}^{dag}H$
$H\bar{v} = \lambda\bar{v}$
1
$\bar{v}^{dag}H = \lambda^*\bar{v}^{dag}$
2
$\bar{v}^{dag}H\bar{v}$
3 inner product

$1: 3 = \bar{v}^{dag}\lambda\bar{v} = \lambda\bar{v}^{dag}\bar{v}$
$2 = \lambda^*\bar{v}^{dag}\bar{v} \Rightarrow$
$\lambda\bar{v}^{dag}\bar{v} = \lambda^*\bar{v}^{dag}\bar{v}$
$\bar{v}^{dag}\bar{v} > 0 \quad \bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \bar{v}^{dag}\bar{v} = |v_1|^2 + |v_2|^2 + |v_3|^2 > 0$
$\Rightarrow (\lambda - \lambda^*)\bar{v}^{dag}\bar{v} = 0$
$\Rightarrow \lambda = \lambda^*$
Real eigenvalue
$if\ H^{dag} = H, then\ \lambda^* = \lambda$

Lets consider λ_1, λ_2
$\lambda_1 \neq \lambda_2$
3 $H\bar{v}_1 = \lambda_1\bar{v}_1$
4 $H\bar{v}_2 = \lambda_2\bar{v}_2$
$\bar{v}_2^{dag}H\bar{v}_1 = \bar{v}_2^{dag}\lambda_1\bar{v}_1 = \lambda_1\bar{v}_2^{dag}\bar{v}_1$
$= \lambda_2\bar{v}^{dag}\bar{v}_1$
$(\lambda_1 - \lambda_2)\bar{v}_2^{dag}\bar{v}_1 = 0$
$\Rightarrow \bar{v}_2^{dag}\bar{v}_1$
Eigenvectors are orthogonal innerproduct =0
Eigenvectors of hermitian matrix with distinct eigenvalues are orthogonal
Normalize eigenvectros
If $\bar{v}_1^{dag}\bar{v}_1 = c$
Then $\bar{w}_1 = \frac{1}{\sqrt{c}}$
is also an eigenvector
And $\bar{w}_1^{dag}\bar{w}_1 = \frac{1}{\sqrt{c}}\frac{1}{\sqrt{c}}\bar{v}_1^{dag}\bar{v}_1 = 1$ is normalized
So
$\bar{v}_1^{dag}\bar{v}_1 + \bar{v}_2^{dag} = \bar{v}_2^{dag}\bar{v}_2 = 1$
$\bar{v}_1^{dag}\bar{v}_2 = 0$
$\bar{v}_1 and\ \bar{v}_2$ form an orthonormal set or a basis
$\bar{v}_i^{dag}\bar{v}_j = \delta_{ij}$
$$\delta_{ij} = \begin{matrix} 1\ if\ i = j \\ 0\ otherwise \end{matrix}$$
Hermetian matrix can be diagonalized by a unitary transformation
In general
$S^{-1}AD = D$
$S^{-1}HS = D$
$$D = \begin{pmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_N \end{pmatrix} = D^{\text{dag}}$$

So

$D^{dag} = D = (S^{-1}HS)^{dag} = S^{dag}H^{dag}S^{-1dag} = S^{dag}HS^{-1dag}$

$\Rightarrow S^{-1}HS = S^{dag}HS^{-1dag}$

$S^{dag} = S^{-1} \Rightarrow S\ is\ unitary$

$U^{dag}HU = D$

All of this holds also for symmetric real matrices

$A, A_{ij} \in \mathbb{R}$

$A^T = A$

$\lambda \in \mathbb{R}$

$\bar{v}_i^T \bar{v}_j = \delta_{ij}$

$O^T AO = D$

Orthogonal transformation

Ex

$H = \begin{pmatrix} 2 & 2-i \\ 2+i & 6 \end{pmatrix} = H^{dag}$

$TrH = 2 + 6 = 8$

$\det H = 2 * 6 - (2+i)(2-i) = 12 - 4 - 1 = 7$

Eigenvalues

$\det(H - \lambda\mathbb{I}) = \begin{vmatrix} 2-\lambda & 2-i \\ 2+i & 6-\lambda \end{vmatrix} = (2-v)(6-\lambda) - (2+i)(2-i)$

$\lambda^2 - 8\lambda + 7 = (\lambda - 7)(\lambda - 1) = 0$

$\lambda_1 = 7, \lambda_2 = 1$

$\lambda_i: H\bar{v}_1 = 7\bar{v}_1 \Rightarrow \begin{pmatrix} 2 & 2-i \\ 2+i & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$

$2 + (2-i)y = 7x: 5x = (2-i)y$

$(2+i)x + 6y = 7y: y = (2+i)x$

$\Rightarrow solution$

$\begin{matrix} x = 1 \\ y = 2+i \end{matrix} \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}, \bar{v}_1^{dag}\bar{v}_1 = (1 \quad 2-i)\begin{pmatrix} 1 \\ 2-i \end{pmatrix} = 1 + 5 = 6$

Normalize

$\bar{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2+i \end{pmatrix}, \begin{matrix} H\bar{v}_1 = 7\bar{v}_1 \\ \bar{v}_1^{dag}\bar{v}_1 = 1 \end{matrix}$

$\lambda_2 = 1: H\bar{v}_2 = 1\ \bar{v}_2$

$\begin{pmatrix} 2 & 2-i \\ 2+i & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{matrix} 2x + (2-i)y = x \\ (2+i)x + 6y = y \end{matrix} \Rightarrow \begin{matrix} x = -(2-i)y \\ 5y = -(2+i)x \end{matrix} \Rightarrow \begin{matrix} x = -2+i \\ y = 1 \end{matrix}$

$\bar{v}_2 = \begin{pmatrix} -2+i \\ 1 \end{pmatrix}$

$\bar{v}_2^{dag}\bar{c}_2 = 1 + 5 = 6$

$\Rightarrow \bar{v}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -2+i \\ 1 \end{pmatrix}$

test

$\bar{v}_2^{dag}\bar{v}_1 = 0?$

$\frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}(-2-i \quad 1)\begin{pmatrix} -2+i \\ 1 \end{pmatrix} = \frac{1}{6}(-2-i+2+i) = 0$

$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2+i}{\sqrt{6}} \\ \frac{2+i}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -2+i \\ -2-i & 1 \end{pmatrix}$

$U^{dag} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 2-i \\ -2-i & 1 \end{pmatrix}$

$U^{dag}U = \frac{1}{(\sqrt{6})^2}\begin{pmatrix} 1 & 2-i \\ -2-i & 1 \end{pmatrix}\begin{pmatrix} 1 & -2+i \\ -2-i & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1+5 & 0 \\ 0 & 1+5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$U^{dag}HU = \cdots = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\begin{array}{c} A = S^{-1}AS = D \\ H^{dag} = H: U^{dag}HU = D \\ A^T = A: O^{dag}AO = D \end{array}}$$

$$e^A = \sum_{n=0}^{\infty} \frac{A^N}{N!} = \frac{1}{0!} + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$A^N = SDS^{-1}SDS^{-1}SDS^{-1} \ldots SDS^{-1} = SD^nS^{-1}$$

$$\Rightarrow \boxed{A = SDS^{-1}}$$

$$D = \begin{pmatrix} f_1 & & & \emptyset \\ & f_2 & & \\ & & \ldots & \\ \emptyset & & & f_n \end{pmatrix}, D^2 = \begin{pmatrix} f_1^2 & & & \emptyset \\ & f_2^2 & & \\ & & \ldots & \\ \emptyset & & & f_n^2 \end{pmatrix}, D^n = \begin{pmatrix} f_1^n & & & \emptyset \\ & f_2^n & & \\ & & \ldots & \\ \emptyset & & & f_n^n \end{pmatrix}$$

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_n S \frac{D^n}{n!} S^{-1} = S\left(\sum_n \frac{D^n}{n!}\right)S^{-1}$$

$$\sum_n \frac{D^n}{n!} = \begin{pmatrix} \sum \frac{f_1^n}{n!} & & & \emptyset \\ & \sum \frac{f_2^n}{n!} & & \\ & & \ldots & \\ \emptyset & & & \sum \frac{f_n^n}{n!} \end{pmatrix}$$

It's because we don't have D^n but $\sum D^N$ you also sum each matrix to get this result

$$= \begin{pmatrix} e^{f_1} & & \emptyset & \\ & e^{f_2} & & \\ & & \ldots & \\ \emptyset & & & e^{f_n} \end{pmatrix}$$

$$e^A = Se^Ds^{-1}$$

$$\det A = \det(Se^DS^{-1}) = \det S \det e^D \det S^{-1} = \det e^D$$

$$\begin{pmatrix} e^{\lambda_1} & & & \emptyset \\ & e^{\lambda_2} & & \\ & & \ldots & \\ \emptyset & & & e^{\lambda_N} \end{pmatrix} = e^D$$

$$e^A = Se^DS^{-1}$$

$$\det e^A = \det Se^DS^{-1} = \det S \det d^D \det S^{-1} = \det e^D$$

$$\det A = \prod_i \lambda_i$$

$$\det e^D = e^{\lambda_1}e^{\lambda_2}e^{\lambda_3} \ldots e^{\lambda_N} = e^{\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_N} = e^{\sum_i \lambda_i} = e^{Tr(A)}$$

$$\det e^D = e^{Tr(A)}$$

Expand

$$e^Ae^B = \left(1 + A + \frac{A^2}{2} + \cdots\right)\left(1 + B + \frac{B^2}{2} + \cdots\right) = 1 + A + B + \frac{A^2}{2} + \frac{B^2}{2} + AB + \cdots$$

$$e^{A+B} = 1 + A + B + \frac{1}{2}(A+B)(A+B) + \cdots$$

$$= 1 + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \cdots$$

They differ!

$$AB \leftrightarrow \frac{1}{2}(AB + BA)$$

$AB \neq BA$!

$$e^Ae^B \neq e^{A+B}$$

$$[A, B] = AB - BA$$

Correct for the mismatch by adding commutators

$$\frac{1}{2}(AB+BA)+\frac{1}{2}[A,B]=AB$$

Campbell-baker-hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{2}[B,[A,B]]+\cdots}$$

$$[B,[A,B]] * B[A,B]-[A,B]B = B(AB-BA)-(AB-BA)B$$

Suppose we have two hermitian N*N matrices

$A = A^{dag}, B = B^{dag}$

When do these have a common set of eigenvectors?

Can be diagonalized simultaneously?

Answer: if A and B commute

$$[A,B] = AB - BA = 0$$

i) $A\bar{v}_i = \lambda_i \bar{v}_i, \lambda_i \; all \; different$

Lets assume AB=BA

$AB\bar{v}_i = BA\bar{v}_i = B\lambda_i\bar{v}_i = \lambda_i B\bar{v}_i$

$\Rightarrow A(B\bar{v}_i) = \lambda(B\bar{v}_i)$

If $\bar{v}_i$ is the eigenvector of A, then so is $B\bar{v}_i$ with the same eigenvalue

Every eigenvector is uniquely determined up to normalization

$B\bar{v}_i \sim \bar{v}_i$

$\Rightarrow B\bar{v}_i = \mu_i \bar{v}_i, \; \mu_i \; scalar$

$\bar{v}_i$ is indeed an eigenvector of B with eigenvalues μ_i

ii) Let's assume

$A\bar{v}_i = \lambda_i \bar{v}_i$

$B\bar{v}_i = \mu_i \bar{v}_i$

Common set of eigenvectors

$\bar{v}_i + \bar{v}_i = \delta_{ij}$

Orthogonal set basis

Every vector can be written as

$$\bar{x} = \sum_i c_i \bar{v}_i$$

$$AB\bar{x} = \sum_i c_i \lambda_i \mu_i \bar{v}_i$$

$$BA\bar{x} = \sum_i c_i \lambda_i \mu_i \bar{v}_i$$

$\Rightarrow AB\bar{x} = BA\bar{x} \forall \bar{x}$

$AB = BA$

Commute!

<u>Degenerate eigenvalues</u>

If $H = H^{dag}$ or $M^T = M$ then eigenvectors form a basis, i.e. they form an orthogonal set

$\bar{v}_i^{dag}\bar{v}_j = \delta_{ij}$

i, j label the eigenvectors

$$\delta_{ij} \begin{matrix} = & 1 \; if \; i = j \\ = & 0 \; if \; i \neq j \end{matrix}$$

This is true when all λ_i's are distinct. If some λ_i's are degenerate (equal), this is correct, but requires still some more work

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$A^T = A; \lambda \in \mathbb{R}$

$Tr\,A = 0$

$\det A = -2 + 0 + 0 - 0 - 0 + 18 = 16$

$$\det(A - \lambda\mathbb{I}) = \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & -2-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = -(\lambda+2)\left((\lambda-1)^2 - 9\right)$$

$= 0$

$\lambda = -2$

$(\lambda-1)^2 - 9 = 0 \Rightarrow (\lambda-1)^2 = 9$

$\lambda - 1 = \pm 3: \lambda = 4, \qquad \lambda = -2$

$\lambda = 4, -2, -2$

Degenerate

$$\lambda = 4: \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{matrix} x + 3z = 4x \\ -2y = 4y \\ 3x + z = 4z \end{matrix} \Rightarrow \begin{matrix} x = z \\ y = 0 \\ x = z \end{matrix} \Rightarrow \bar{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \bar{v}^{dag} \bar{v} = 1$$

$$\lambda = -2: \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{matrix} x + 3z = -2x \\ -2y = -2y \\ 3x + z = -2z \end{matrix} \Rightarrow \begin{matrix} x = -z \\ y = y \\ x = -z \end{matrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ b \\ -a \end{pmatrix}$$

All eigenvectors

Chose <u>two</u> eigenvectors and make them orthogonal

$$\Rightarrow \bar{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \bar{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Now

$$\bar{v}_i^{dag} \bar{v}_j = \delta_{ij}$$

$$\bar{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} is\ eigenvector$$

But not orthogonal to $\bar{v}_2$

☆ BB: sheet with algebra

Vector calculus and Integration

01 November 2011
14:46

2d 3d integrals-> integral theorem
Conservative vector fields => em maxwell equations

$$\int_a^b f(x)dx$$

Divide interval a<x<b in N subintervals of length Δx such that $b - a = N\Delta x$
In each interval pick a point x; ;=1,…,N
Assume f(x) is constant in each subinterval
Add together the area for each subinterval

$$\sum_{i=1}^{N} \Delta x f(x_i)$$

This becomes a better approximation as $\Delta x \to 0$
Definition

$$\int_a^b f(x)dx = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta x\, f(x_i)$$

With

$$N\Delta x = b - a \Rightarrow \Delta x \to 0\; if\, \frac{b-a}{N}, N \to \infty$$

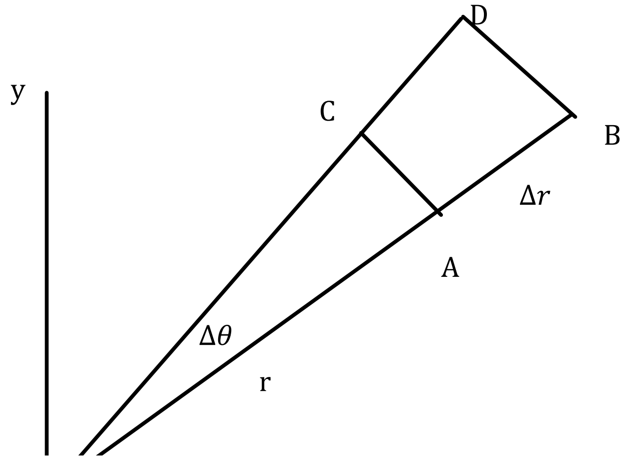
$$\boxed{\int_a^b f(x)dx = \lim_{N\Rightarrow\infty} \sum_{i=0}^{N} f(x_i)\Delta x}$$

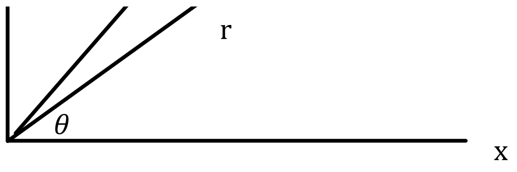
$$\int\int dx\, dy\, xy = \int_0^1 \int_0^{\sqrt{1-x^2}} xy = \frac{1}{8}$$

$$for\; x^2 + y^2 = 1$$

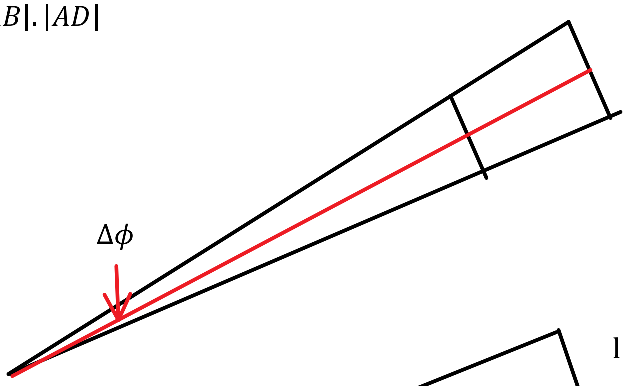
$r = \sqrt{x^2 + y^2}$
$x = r\cos\theta$
$y = r\sin\theta$
$0 < r < 1$
$0 < \theta < \frac{\pi}{2}$

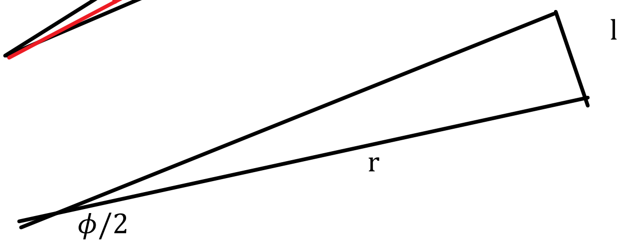
$$\int\int dx\, dy\, f(x,t) \Rightarrow \int_0^1 dr \int_0^{\frac{\pi}{2}} d\theta \ldots$$





$|AB|.|AD|$





$|AB| = \Delta r$
$|AD| = ?$

$l = r\sin\left(\frac{\Delta\phi}{2}\right)$

$|AD| = 2l = 2r\sin\left(\frac{\Delta\phi}{2}\right)$

$\Delta\phi \to 0$

$|AD| \to 2r\frac{\Delta\phi}{2} = r\Delta\phi$

$\phi \to \theta$

$|AB||AD| = \Delta r\, r\Delta\phi$
$= r\,\Delta r\,\Delta\theta$

Jacobian:

$J = r$

For polar coordinates

$\int\int dx\, dy\, f(x,t) = \int\int dr\, d\theta\;\; r\, f(r\cos\theta\,, r\sin\theta)$

$\int\int dxdy 1 = area$

$\int_0^a dr \int_0^{2\pi} d\theta\, r\, 1 = \int_0^a dr\, r \int_0^{2\pi} d\theta = \frac{1}{2}r^2\Big|_0^a\, \theta\Big|_0^{2\pi} = \frac{a^2}{2}2\pi = \pi a^2$

$\int\int dxdy\; xy$

$-\int_0^1 dr \int_0^{\frac{\pi}{2}} d\theta\, r\, r\cos\theta\, r\sin\theta$

$\int_0^1 dr\, r^3 \int_0^{\frac{\pi}{2}} d\theta\cos\theta\sin\theta = \left(\frac{1}{4}r^4\Big|_0^1\right)\left(-\frac{1}{2}\cos^2\theta\Big|_0^{\frac{\pi}{2}}\right) = \frac{1}{4} * \left(-\frac{1}{2}(0-1)\right) = \frac{1}{8}$

$\frac{d}{d\theta}(\cos^2\theta) = 2\cos\theta\frac{d\cos\theta}{d\theta} = -2\cos\theta\sin\theta$

$\frac{d}{d\theta}(\sin^2\theta) = 2\sin\theta\frac{d\sin\theta}{d\theta} = 2\sin\theta\cos\theta$

$\int\int dxdy\,\sqrt{x^2+y^2}$

From 1 to 2

$$\int_1^2 dr \int_0^\pi d\theta\, rr = \int_1^2 dr\, r^2 \int_0^\pi d\theta = \frac{1}{3} r^3 \Big|_1^2 \pi = \frac{\pi}{3}(8-1) = \frac{7\pi}{3}$$

<u>3d integrals</u>

$\int\int\int dxdydz\, f(x,y,z)$

Ex

$f = 1$

$\int\int\int dxdydz\, 1 = volume\ of\ v$

Ex

$f(x,y,z) = n(x,y,z)$

Mass density $\left[\frac{kg}{m^3}\right]$

$$\int\int\int_v dxdydz\, n(x,y,z) = M = total\ mass\ inside\ volume$$

$$\int_0^a dx \int_0^a dy \int_0^a dx\, 1$$

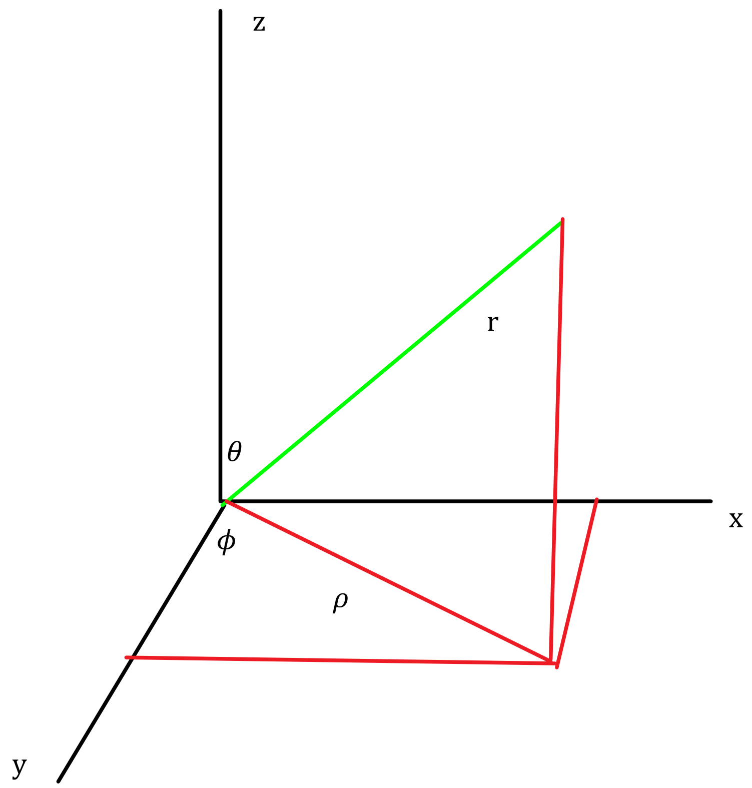
$$= x\Big|_0^a y\Big|_0^a z\Big|_0^a = a * a * a = a^3$$

Sphere radius a volume

$\frac{4}{3}\pi$^3

Rotational symmetry

⇒ spherical coordinates



Last time: 2d, 3d integral

Rotational symmetry

2D:

Polar coordinates

$x = r\cos\phi$

$y = r\sin\phi$
$r = \sqrt{x^2 + y^2}$

$\int\int dxdy\, f(x, y)$
$= \int dr \int d\phi\, r\, f(r\cos\phi, r\sin\phi)$
Inc jacobian r
$-\infty < x, y < \infty$
$0 < r < \infty$
$0 < \phi < 2\pi$
$y > 0 \leftrightarrow 0 < \phi < \pi$

3D

$\bar{r} = (x, y, z)$
$r, \phi\theta$
$\rho =$ projection in xy-plane
$\rho = r\sin\theta$

$x = r\cos\phi\sin\theta$
$y = r\sin\phi\sin\theta$
$x = r\cos\theta$
Spherical coordinates
$-\infty < x, y, z < \infty$
$0 < r < \infty$
$0 < \phi < 2\pi$
$0 < \theta < \pi$

$\int\int\int dxdydz f(x, y, z)$
$\int dr \int d\phi \int d\theta J f(x, y, z)$
J=jacobian "change of variables

Jacobian for spherical coordinates

$$\boxed{J = r^2\sin\theta}$$

Ex
Sphere
$0 < r < a$
$0 < \phi < 2\pi$
$0 < \theta < \pi$
Volume of this sphere

$$\int_0^a dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2\sin\theta$$

$$\int_0^a dr\ r^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta$$

$$\left[\frac{1}{3}r^3\right]_0^a [\phi]_0^{2\pi} [-\cos\theta]_0^{\pi}$$

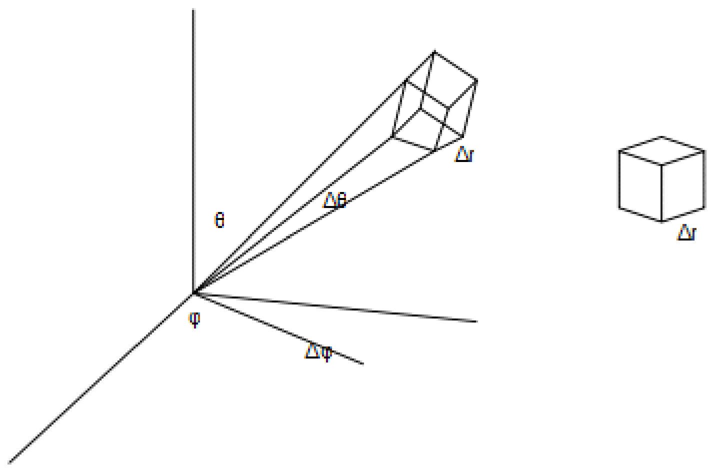
$$= \frac{a^3}{3} * 2\pi * (--1--1)$$

$$= \frac{4\pi}{3}a^3$$

Hemisphere

$$0 < \theta < \frac{\pi}{2}$$

Jacobian is the elementary volume, obtained by changing
$r \to r + \Delta r$
$\theta \to \theta + \Delta\theta$
$\phi \to \phi + \Delta\phi$



$$\begin{bmatrix} in\ cartesian\ coordinates \\ \Delta x \Delta y dz \end{bmatrix}$$

$r \to r + \Delta r$
One side Δr

$\theta \to \theta + \Delta\theta$

$$2r \sin\left(\frac{\Delta\theta}{2}\right)$$

$\to r\Delta\theta$

$\Delta r\ r\Delta\theta\ r \sin\theta\ \ \Delta\phi$
$= r^2 \sin\theta\, \Delta r\ \Delta\theta\Delta\phi$
$r^2 \sin\theta = jacobian$

Partial differentiation

16 November 2011
11:04

1 variable x, f(x)
2 variables f(x,t)
Partial derivaties

$$\frac{\delta f(x,y)}{\delta x}, y\ is\ fixed$$

$$\frac{\delta f(x,y)}{\delta x}, x\ is\ fixed$$

3 2nd derivatives

$$\frac{\delta^2 f}{\delta x^2}, \frac{\delta^2 f}{\delta y^2}, \frac{\delta^2 f}{\delta x \delta y} = \frac{\delta^2 f}{\delta y \delta x}$$

Ex $f(x,y) = x^3 y + y^2 \sin x$

$$\frac{\delta f}{\delta x} = 3x^2 y + y^2 \cos x$$

$$\frac{\delta f}{\delta y} = x^3 + 2y \sin x$$

$$\frac{\delta^2 f}{\delta x^2} = 6xy - y^2 \sin x$$

$$\frac{\delta^2 f}{\delta y^2} = 2 \sin x$$

$$\frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta}{\delta x}\left(\frac{\delta f}{\delta y}\right) = 3x^2 + 2y \cos x = \frac{\delta^2 f}{\delta x \delta y}$$

Change of variables
1D

$$\int dy f(x)$$

x=x(u)

$$dx = \left(\frac{\delta x}{\delta u}\right) du$$

$$\int du \left(\frac{\delta x}{\delta u}\right) f\big(x(u)\big)$$

$$\frac{\delta x}{\delta u} \rightarrow jacobian$$

Ex

$$\int_0^{\sqrt{\pi}} dx\ x \sin x^2$$

$$x = \sqrt{u}$$

$$dx = \frac{\delta x}{\delta u} du = \frac{1}{2\sqrt{u}} du$$

$$= \int_0^{\pi} du \frac{1}{2\sqrt{u}} \sqrt{u} \sin u$$

$$= \int_0^{\pi} du \frac{1}{2} \sin u$$

$$= -\frac{1}{2} \cos u \Big|_0^{\pi} = 1$$

Change of variables in more dimensions
(x,y,z)→(u,v,w)
Ie

x=x(u,v,w)
y=y(u,v,w)

z=z(u,v,w)

Here

$\frac{\delta x}{\delta u}, \frac{\delta x}{\delta v}$

$\frac{\delta y}{\delta u}$

Etx

9 different partial derivatives

Jacobian= absolute value of

$$\det\begin{pmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} & \frac{\delta x}{\delta w} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} & \frac{\delta y}{\delta w} \\ \frac{\delta z}{\delta u} & \frac{\delta z}{\delta v} & \frac{\delta z}{\delta w} \end{pmatrix} = \frac{\delta(x,y,z)}{\delta(u,v,w)}$$

$$\iiint dxdydz\, f(x,y,z) = \iiint dudvdw \left|\frac{\delta(x,y,z)}{\delta(u,v,w)}\right| f\big(x(u,v,w), y(⬚), z(⬚)\big)$$

Ex 2dd polar coordinates

$x = x(r\,,\phi) = r\cos\phi$

$y = y(r,\phi) = r\sin\phi$

$$\frac{\delta(x,y)}{\delta(r,\phi)} = \det\begin{pmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \phi} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \phi} \end{pmatrix} = \det\begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} = r\cos^2\phi + r\sin^2\phi = r$$

Ex 3d Spherical coordinates

$(x,y,z) \to (r,\phi,\theta)$

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

$c\phi = cos\phi$

$s\theta = \sin\theta$

$$\frac{\delta(x,y,z)}{\delta(r,\phi,\theta)} = \det\begin{pmatrix} c\phi s\theta & -rs\phi s\theta & rc\phi c\theta \\ s\phi s\theta & rc\phi s\theta & rs\phi c\theta \\ c\theta & 0 & -rs\theta \end{pmatrix} = \pm r^2\sin\theta$$

$$J = \left|\frac{\delta(x,y,z)}{\delta(r,\phi,\theta)}\right| = r^2\sin\theta$$

Ex $\int_P dx\, dy\, xy$

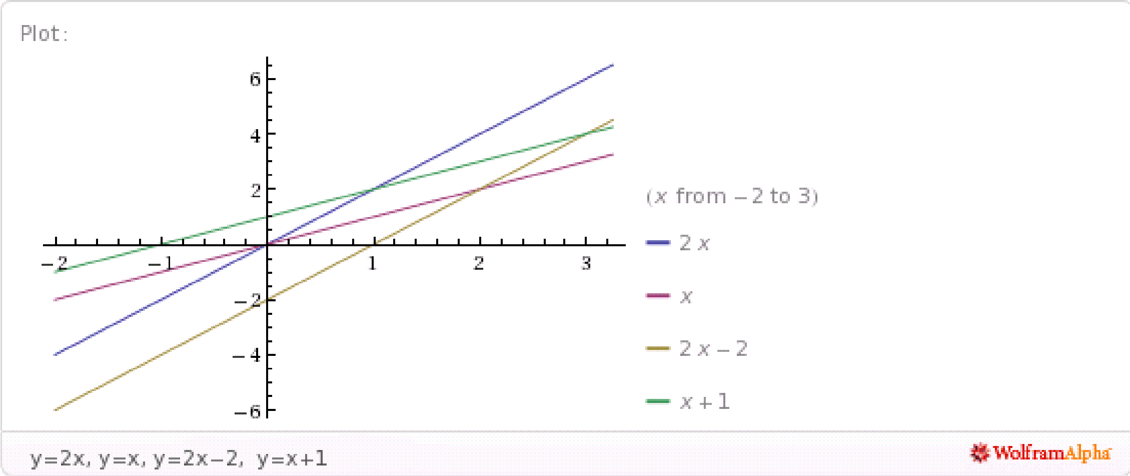
p=bounded region

y=2x,

y=x,

y=2x-2,

y=x+1



Change of variables
x=u-v
y=2u-v

$$\det\begin{pmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{pmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1 + 2 = 1$$

$\iint du\, dv\, 1(u - v)(2u - v)$

Boundaries

$y = 2x: 2u - v = 2(u - v): v = 0$

$y = 2x - 2: 2u - v = 2u - 2v - 2: v = -2$

$y = x: 2u - v = u - v: u = 0$

$y = x + 1: 2u - v = u - v + 1: u = 1$

$-2 < v < 0$

$0 < u < 1$

$$\int_{-2}^{0} dv \int_{0}^{1} du\, (u - v)(2u - v)$$

$$= \int_{-2}^{0} dv \int_{0}^{1} du\, (2u^2 - 3uv + v^2) = 7$$

Line integrals

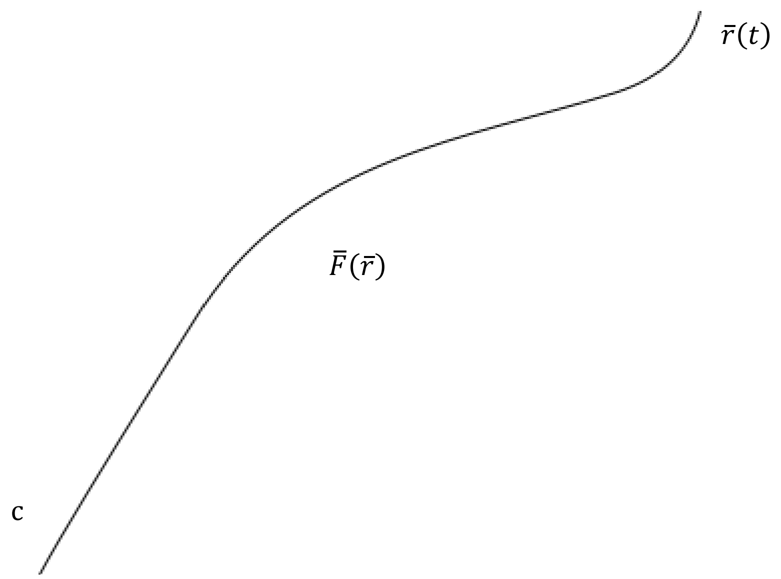
16 November 2011
11:38

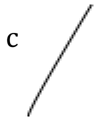
Work done by force
Particle $m\bar{a} = \bar{F}$
3d:$\bar{F}(\bar{r}) = \left(F_x(\bar{r}), F_y(\bar{r}), F_z(\bar{r})\right)$
    Vector field
$Work = distance * force$
Ex simplest
    Particle moves in straight line
    Constant force
        $\bar{F} = \left(F_x, F_y, F_z\right)$
    Displacement
        $\bar{r}_2 - \bar{r}_1$
Work:
    $\bar{F} * (\bar{r}_2 - \bar{r}_1)$
    $= F_x(x_2 - x_1) + F_y(y_2 - y_1) + F_z(z_2 - z_1)$

Curved path
    Nonconstant force

    Consider a small interval
    $\bar{r} \to \bar{r} + \Delta\bar{r}$
    $\bar{F}(\bar{r})$ is approximately constant
    $W_{\Delta\bar{r}} \approx \bar{F}(\bar{r}) * \Delta\bar{r}$
    Add all small contributions together
    Total work
    $$W = \sum_{i=1}^{N} F(\bar{r}_i)\Delta\bar{r}_i$$
    $N \to \infty$
    $$W = \int_c \bar{F}(\bar{r})d\bar{r}$$
    Line integral

    Work:
    $$\int_c \bar{F}(\bar{r})d\bar{r} = W$$





Mechanics

$\bar{r}(t)$

t=time

$\bar{F}(\bar{r})$

$$\bar{F} = m\bar{a} = m\,\frac{d^2\bar{r}}{dt^2}$$

$$\bar{a} = \frac{d\bar{v}}{dt}$$

$$\bar{v} = \frac{d\bar{r}}{dt}$$

$$w = \int \bar{F} d\bar{r}$$

$$= m\int \frac{d^2\bar{r}}{dt^2} d\bar{r}$$

$$\bar{r}(t) \rightarrow t$$

Jacobian:

$$d\bar{r}(t) = \frac{d\bar{r}}{dt} dt$$

$$W = m\int dt\,\frac{d\bar{r}}{dt} * \frac{d^2\bar{r}}{dt^2} = m\int dt\;\bar{v}\frac{d}{dt}\bar{v}$$

$$\bar{v}\frac{d}{dt}\bar{v} = \frac{1}{2}\frac{d}{dt}(\bar{v}^2)$$

$$= m\int dt\,\frac{d}{dt}\left(\frac{1}{2}\bar{v}^2\right) = \frac{1}{2}m\bar{v}^2\,\Big|_{t_{initial}}^{t_{final}} = \Delta kinetic\ energy = work$$

$$\int dt\,\frac{d}{dt} \Leftarrow total\ derivative$$

$$\int_{\bar{\sigma}(t)} \bar{F}(\bar{r})d\bar{r} = \int_{t_i}^{t_f} dt\;\frac{d\bar{\sigma}(t)}{dt} * \bar{F}\big(\bar{\sigma}(t)\big)$$

$$\bar{\sigma}(t) = \big(\sigma_x(t), \sigma_y(t), \sigma_z(t)\big)$$

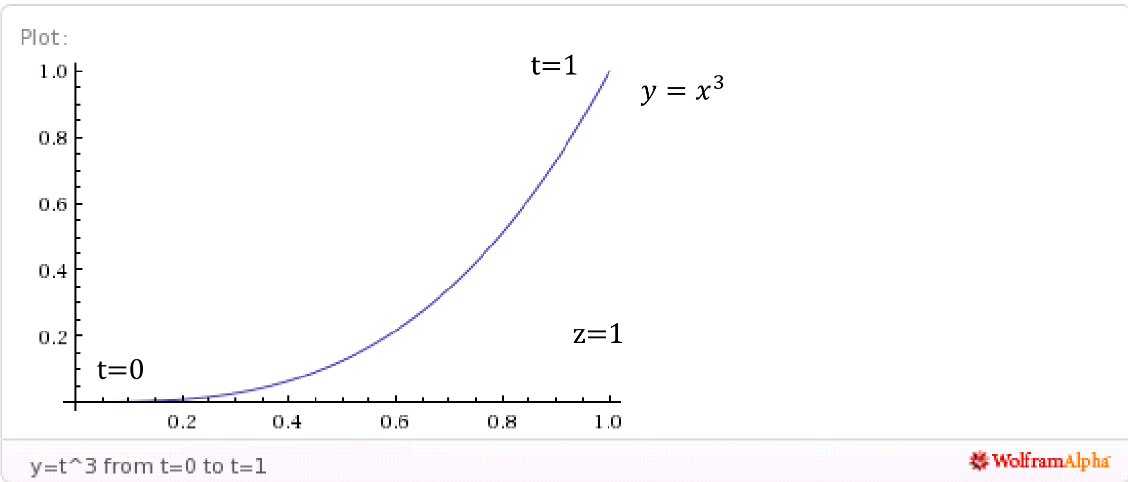
Path in 3d space

t=time, paramiterisation of path

Ex

$$\bar{F}(\bar{r}) = (x^2, z, 3x + 2y)$$

$$\bar{\sigma}(t) = (t, t^3, 1), 0 < t < 1$$



$$\int \bar{F} d\bar{r} = \int_0^1 dt\,\frac{d\bar{\sigma}}{dt} * \bar{F}\big(\bar{\sigma}(t)\big)$$

$$\frac{d\bar{\sigma}}{dt} = (1.3t^2, 0)$$
$$\bar{F}(\bar{\sigma}(t)) = (t^2, 1{,}3t + 2t^3)$$
$$x \to t, y \to t^3, z \to 1$$
$$\int \frac{d\bar{\sigma}}{dt}\bar{F} = 1 * t^2 + 3t^2 * 1 \pm (3t + 2t^3)$$
$$= 4t^2$$
$$\Rightarrow \int_0^1 dt\, 4t^2 = \frac{4}{3}t^3\Big|_0^1 = \frac{4}{3}$$

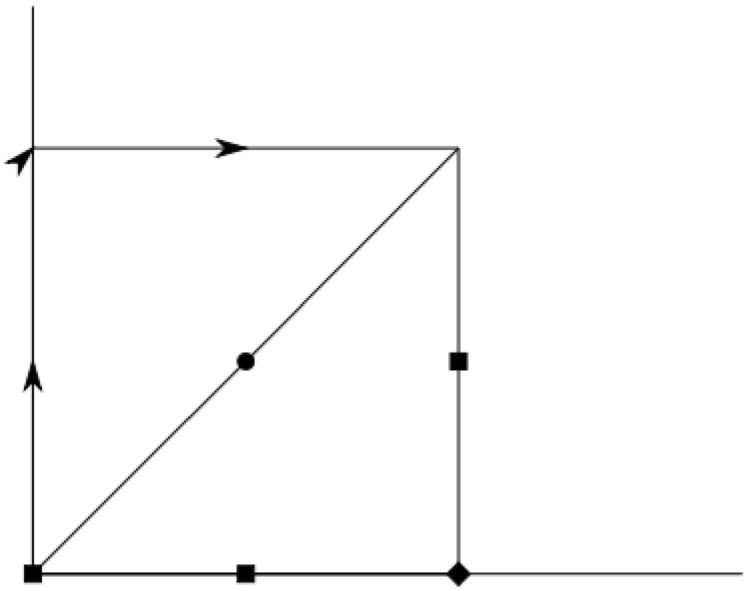
2d example,

$$\bar{F}(x,y) = (2xy, x^2 + axy) = (F_x, F_y)$$

c_1: (0,0) → (1,0) → (1,1)

c_2: (0,0) → (1,1) along diagonal, y=x

c_3: (0,0) → (0,1) → (1,1)



$$W_1 = \int d\bar{r} * \bar{F}(x,y) = \int dx F_x(x,y) \int dy F_y(x,y)$$
$$\bar{F} = (F_x, F_y), d\bar{r} = (dx, dy)$$
$$a(0,0) \to (1,0): 0 < x < 1, y = 0 \Rightarrow dy = 0$$
$$W_{1a} = \int_0^1 dx\, F_x(x,0) + 0 = \int_0^1 dx (2xy)_{y=0} = \int_0^1 dx\, 0 = 0$$
b(1,0) → (1,1)$x = 1, 0 < y < 1, dx = 0$
$$W_{1b} = 0 + \int_0^1 dy\, F_y(1,y) = \int_0^1 dy\,(1 + ay) = y + \frac{1}{2}ay^2|_0^1 = 1 + \frac{a}{2}$$
$$W_1 = 1 + \frac{a}{2}$$

C_2

$$y = x, dy = dx$$
$$\frac{dy}{dx} = 1, y(x) = x$$
$$W_2 = \int d\bar{r}\bar{F}(\bar{r}) = \int dx\, F_x(x,y) + \int dy F_{y(x,y)}$$
$$= \int_0^1 dx\, F_x(x,x) + \int_0^1 dx\, F_y(x,x)$$
$$= \int_0^1 dx\big(F_x(x,x) + F_y(x,x)\big) = \int_0^1 dx(2x^2 + x^2 + ax^2)$$

$$\int_0^1 dx(3+a)x^2 = \frac{3+a}{3}x\hat{}2\Big|_0^1$$
$$= 1 + \frac{a}{3} = W_2$$

C_3

$W_{3a}: x = 0, 0 < y < 1, dx = 0$
$$\int_0^1 dy\, F_y(0,y) = \int_0^1 dy\, 0 = 0$$
$W_{3b}: 0 < x < 1, y = 1, dy = 0$
$$\int_0^1 dx\, F_x(x,1) = \int_0^1 dx\, 2x = x^2\Big|_0^1 = 1$$
$W_3 = 1$

$W_1 = 1 + \frac{a}{2}$
$W_2 = 1 + \frac{a}{3}$
$W_3 = 1$
Work done depends on the path that is taken
If $a = 0: W_1 = W_2 = W_3$
$\bar{F}(x,y) = (2xy, x^2)$
$\bar{F} = -\nabla\phi = \left(-\frac{\delta\phi}{\delta x}, -\frac{\delta\phi}{\delta y}\right)$
$\phi(x,y) = -x^2y + constant$
$\frac{\delta\phi}{\delta x} = -2xy = -F_x(x,y)$
$\frac{\delta\phi}{\delta y} = -x^2 = -F_y(x,y)$
$\phi = potential$
When
$a \neq 0, \phi$ does not exist!

Ex

$y^2 = x^3$
$y - x^{\frac{3}{2}} = x\sqrt{x}$
$A = (1,1) B = (2, 2\sqrt{2})$
$\bar{F}(x,y) = (xy, x)$

$$W = \int_c d\bar{r}\bar{F}(\bar{r}) = \int dt \frac{d\bar{\sigma}}{dt} * \bar{F}(\bar{\sigma}(t))$$
What is $\bar{\sigma}(t)$?
$\bar{\sigma}(t): y = t^3, x = t^2$
$y^2 = t^6$
$x^3 = t^6$

$\bar{\sigma}(t) = (t^2, t^3)$
$A = (1,1): t = 1$
$B = (2, 2\sqrt{2}): t = \sqrt{2}$
$\frac{d\bar{\sigma}}{dt} = (2t, 2t^2)$
$\bar{F}(\bar{\sigma}(t)) = (t^5, t^2)$
$x = t^2$
$y = t\%3$

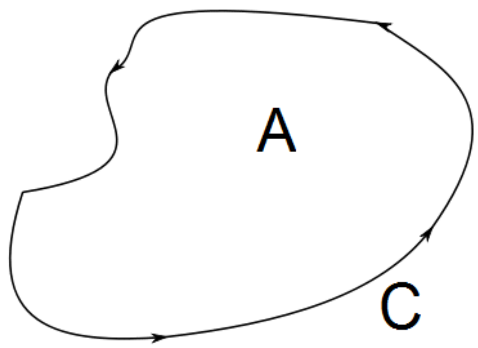
$\bar{\sigma}' * \bar{F} = 2tt^5 + 3t^2t^2$
$= 2t^6 + 3t^4$

$$W = \int_1^{\sqrt{2}} dt\ (2t^6 + 3t^4) = 2/$$

Stokes theorem (2D)

$$\oint_C \bar{F} d\bar{r} = \iint_A dxdy \left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} \right)$$

Sometimes LHS or RHS is easier to compute



$$\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} = 0, then \oint_c \bar{F} d\bar{r}$$

Vanishes for all closed contours C

Be derived from a potential ϕ

$$F_x = -\frac{\delta}{\delta x}\phi, F_y = -\frac{\delta}{\delta y}\phi$$

Then

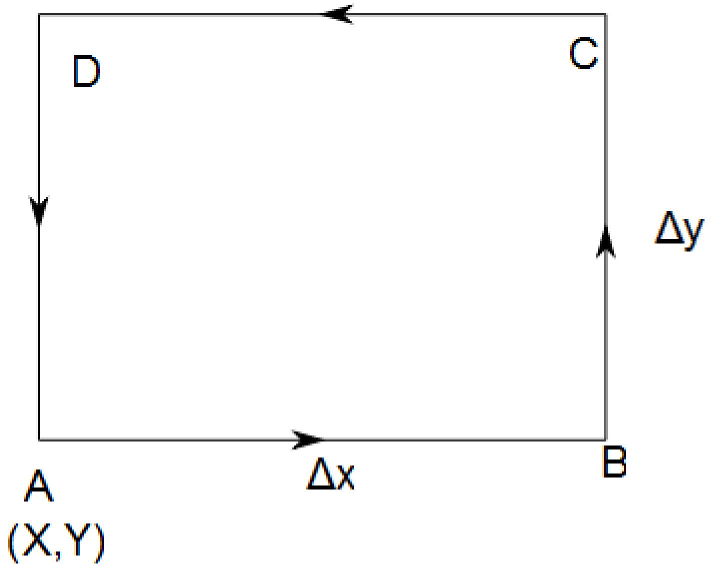
$$\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}$$

$$= \frac{\delta}{\delta x}\left(-\frac{\delta}{\delta y}\phi\right) - \frac{\delta}{\delta y}\left(-\frac{\delta}{\delta x}\phi\right)$$

$$= \frac{\delta^2}{\delta x \delta y}\phi - \frac{\delta^2}{\delta y \delta x}\phi = 0$$

These vector fields are called conservative

Proof:



Evaluate

$$\oint_{ABCD} \bar{F} d\bar{r}$$

1. AB: $y = Y, X < x < X + \Delta x$
$$\int_{AB} \bar{F} d\bar{r} = \int_{X}^{X+\Delta x} dx\, F_x(x, Y)$$
2. BC: $x = X + \Delta X, Y < y < Y + \Delta y$
$$\int_{BC} \bar{F} d\bar{r} = \int_{Y}^{Y+\Delta y} dy F_y(X + \Delta x, y)$$
3. CD: $X + \Delta x > x > X, y = Y + \Delta y$
$$\int_{CD} \bar{F} d\bar{r} = \int_{X+\Delta x}^{X} dx\, F($$
4. DA: $Y + \Delta y > y > Y, x = X$
$$\int_{CD} \bar{F} d\bar{r} = \int_{Y+\Delta y}^{y} F_y(X, y)$$

so

$$\oint \bar{F} d\bar{r} = \int_{AB} \square + \int_{CB} \square + \int_{BC} \square + \int_{DA} \square$$

$$= \int_{X}^{X+\Delta x} (F_x(x, Y) - F_x(x, Y + \Delta Y)) + \int_{Y}^{Y+\Delta y} \left(F_y(X + \Delta x, y) - F_y(X, y)\right)$$

Use $\Delta x, \Delta y \ll 1$, use

$$\int_{X}^{X+\Delta x} dx\, f(x) \approx f(X)\Delta x$$

$$\Rightarrow [F_x(X, Y) - F_x(X, Y + \Delta y)]\Delta x + \left[F_y(X + \Delta x, y) - F_y(X, Y)\right]\Delta y$$

Derivative

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\Rightarrow -\frac{\delta}{\delta y} F_x(X, Y)\Delta y \Delta x + \frac{\delta}{\delta x} F_y(X, Y)\Delta x \Delta y$$

$$\oint_{ABCD} \bar{F} d\bar{r} = \left(\frac{\delta}{\delta x} F_y - \frac{\delta}{\delta y} F_x\right) \Delta x \Delta y = \iint dxdy \left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$$

$$\Delta x \Delta y = area$$

Proved for small square

General case- divide into many small squares

$$area = \sum area\ of\ squares$$

Ex

$$\bar{F}(\bar{r}) = (2xy, x^2 + axy)$$

$$W = \int_{AB} \bar{F} d\bar{r} = 1 \; for \; all \; paths \; if \; a = 0$$

1) Closed contour

$$\int_A^B dx\, f(x) = -\int_B^A dx\, f(x)$$

2) Is $\bar{F}$ conservative?

$$\bar{F} = 0\nabla\phi$$

$$\phi(x,y) = -x^2 y + constant$$

3) Stokes

$$\oint \bar{F} d\bar{r} = \iint \left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right) dxdy$$

$$\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} = 2x - 2x = 0$$

(a=0)

$$\Rightarrow If \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} = 0$$

Then

$$\oint_C \bar{F} d\bar{r} = 0 \forall C$$

And

$$\int_{AB} \bar{F} d\bar{r}$$

Depends on begin-end point, not actual contour

NOTE: every vector field

$$\bar{F} = -\nabla\phi$$

Is conservative

$$\frac{\delta}{\delta x} F_y - \frac{\delta}{\delta y} F_x = \frac{\delta}{\delta x}\left(-\frac{\delta}{\delta y}\phi\right) - \frac{\delta}{\delta y}\left(-\frac{\delta}{\delta x}\phi\right) = 0$$

<u>Conservative Force Fields (2D)</u>

$\bar{F}(\bar{r})$ is conservative:

- $\frac{\delta F_x}{\delta y} = \frac{\delta F_y}{\delta x}$
- $\bar{F}(\bar{r}) - \nabla\phi(\bar{r})$

  Or

  $$F_x(\bar{r}) = -\frac{\delta}{\delta x}\phi(\bar{r})$$

  $$F_y(\bar{r}) = -\frac{\delta}{\delta y}\phi(\bar{r})$$

- $\oint_c \bar{F}(\bar{r}) d\bar{r} = 0$

  Follows from stoke's theorem

  $$\oint_c \bar{F}(\bar{r}) d\bar{r} = \iint dxdy \left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$$

- $\int_{AB} \bar{F}(\bar{r}) d\bar{r}$

  Depend only on begin and end points, not actual contour

  $$\int_{C1+C2} \bar{F} d\bar{r} = 0$$

  $$\int_{C1} \bar{F} d\bar{r} + \int_{C2} \bar{F} d\bar{r} = 0$$

  $$\int_{C1} \bar{F} d\bar{r} = \int_{-C2} \bar{F} d\bar{r}$$

3D

$\bar{F}(x,y,z) = \left(F_x(\bar{r}), F_y(\bar{r}), F_z(\bar{r})\right)$

Scalar potential

$\phi(F) = \phi(x,y,z)$

If $\bar{F}(\bar{r}) = -\nabla\phi$

$$F_x = -\frac{\delta\phi}{\delta x}$$

$$F_y = -\frac{\delta\phi}{\delta y}$$

$$F_z = -\frac{\delta\phi}{\delta y}$$

Then $\bar{F}$ is conservative

Cross product

$\bar{F} is\ conservative \Leftrightarrow \nabla \times \bar{F} = 0$

∇=nabla operator

Vector

$$\nabla = \left(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}\right)$$

$\nabla \times \bar{F}\ vector, curl\ of\ \bar{F}_1, curl\ \bar{F}$

$$\nabla \times \bar{F} = \left(\frac{\delta}{\delta y}F_z - \frac{\delta}{\delta z}F_y, \frac{\delta}{\delta z}F_x - \frac{\delta}{\delta x}F_z, \frac{\delta}{\delta x}F_y - \frac{\delta}{\delta y}F_x\right)$$

$\nabla\phi(\bar{r}) = vector$

$$= \left(\frac{\delta\phi}{\delta x}, \frac{\delta\phi}{\delta y}, \frac{\delta\phi}{\delta z}\right)$$

Gradient of ϕ

Grad ϕ

$\nabla * \nabla = scalar$

$$\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$$

$\nabla^2\phi\ scalar$

$$= \frac{\delta^2}{\delta x^2}\phi + \frac{\delta^2}{\delta y^2}\phi + \frac{\delta^2}{\delta z^2}\phi$$

$\nabla^2\bar{F}\ vector$

$$= \left(\nabla^2 F_x, \nabla^2 F_y, \nabla^2 F_z\right)$$

$\nabla.\bar{F} scalar, divergence\ of\ \bar{F}$

$div\ \bar{F}$

$$= \frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z}$$

$$\bar{F}.\nabla = F_x\frac{\delta}{\delta x} + F_y\frac{\delta}{\delta y} + F_z\frac{\delta}{\delta z}$$

$\bar{F}.\nabla\phi$ scalar

$\bar{F}.\nabla\bar{A}\ vector$

$$= \left(\bar{F}\nabla A_x, \bar{F}\nabla A_y, \bar{F}\nabla A_z\right)$$

Ex

$$\phi(\bar{r}) = -\frac{1}{2}x^2y^2z^2 - 2xy + 3$$

$\bar{F} = -\nabla\phi$

$$F_x = -\frac{\delta\phi}{\delta x} = xy^2z^2 + 2y$$

$$F_y = -\frac{\delta\phi}{\delta y} = x^2yz^2 + 2x$$

$$F_z = -\frac{\delta\phi}{\delta z} = x^2y^2z$$

Conservative

$\nabla \times \bar{F} = 0?$

$$\nabla \times \bar{F} = (2x^2yz - 2x^2yz - 2xy^2z, 2xy^2z - 2xy^2z, 2xyz^2 + 2 - 2xyz^2 - 2) = (0,0,0)$$

$\bar{F}$ is conservative

Ex Vector field $\bar{F}$ follows from potential $\phi(\bar{r}) = \frac{1}{r}$

$r = \sqrt{x^2 + y^2 + z^2}$

$\bar{F} = -\nabla\phi = \left(\frac{\delta\phi}{\delta x}, \frac{\delta\phi}{\delta y}, \frac{\delta\phi}{\delta z}\right)$

$\phi = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$

$\frac{\delta\phi}{\delta x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} 2x$

$= -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = -\frac{x}{r^3}$

$\frac{\delta\phi}{\delta y} = -\frac{y}{r^3}$

$\frac{\delta\phi}{\delta y} = -\frac{z}{r^3}$

$\bar{F} = -\nabla\phi$

$= \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$

$\bar{r} = (x, y, z)$

$= \frac{\bar{r}}{r^3} = \frac{1}{r^2}\hat{r}$

$\hat{r} = \frac{\bar{r}}{r} = unit\ vector$

$\hat{r}.\hat{r} = 1$

$\bar{F} = \frac{1}{r^2}\hat{r}$

Strength drops as $\frac{1}{r^2}$

Points in the radial direction

Newton gravity

Coulomb, EM

ϕ is constant on spheres with fixed radius

$\bar{F}$ is perpendicular to the quipotential surface

General statement

$\bar{F} \perp$ surface of constant potential

$\phi(\bar{r}) = \phi(\bar{r} + \delta\bar{r})$

$\delta\bar{r} \ll 1$

$\phi(\bar{r} + \delta\bar{r}) = \phi(\bar{r}) + \delta\bar{r}.\ \nabla\phi(\bar{r}) + \theta(\delta\bar{r}^2)$

$f(x + \Delta x) = f(x) + \Delta x f'(x) + \theta(\Delta x^2)$

$\delta\bar{r} = (\delta x, \delta y, \delta z)$

$\delta x \frac{\delta\phi}{\delta x} + \delta y \frac{\delta\phi}{\delta y} + \delta z \frac{\delta\phi}{\delta z}$

$\phi(\bar{r} + \delta\bar{r}) = \phi(\bar{r}) + \delta\bar{r}\nabla\phi + \cdots$

But

$\phi(\bar{r} + \delta\bar{r}) = \phi(\bar{r})$

Since equipotential surface

$\Rightarrow \delta\bar{r}\nabla\phi = 0 \Rightarrow \delta\bar{r}\bar{F}(\bar{r}) = 0$

$\Rightarrow \delta\bar{r} \perp \bar{F}(\bar{r})$

Stokes theorem (3D)

2D:

$\oint_C \bar{F} d\bar{r} = \iint_A dxdy \left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$

$\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right) = 3rd\ component\ of\ \nabla \times \bar{F}$

$= (\nabla \times \bar{F})_z$

Normal vector of surface

$\hat{n}$

$\hat{n}$ is always $\perp$ to surface, $\hat{n}.\hat{n} = 1$

Surface is in xy-plane $\hat{n} = \hat{z} = (0,0,1)$

$(\nabla \times \bar{F})_z = (\nabla \times \bar{F}) * \hat{n}$

$\hat{n} dxdy = \hat{n} dS = d\bar{S}$

3D

$\oint_C \bar{F} d\bar{r} =$

$\iint d\bar{S} * (\nabla \times \bar{F})$

30 November 2011
09:35

Conservative Force Fields (2D)

$\bar{F}(\bar{r})$ is conservative:

- $\frac{\delta F_x}{\delta y} = \frac{\delta F_y}{\delta x}$
- $\bar{F}(\bar{r}) - \nabla\phi(\bar{r})$

  Or

  $F_x(\bar{r}) = -\frac{\delta}{\delta x}\phi(\bar{r})$

  $F_y(\bar{r}) = -\frac{\delta}{\delta y}\phi(\bar{r})$

- $\oint_c \bar{F}(\bar{r})d\bar{r}$=0

  Follows from stoke's theorem

  $\oint_c \bar{F}(\bar{r})d\bar{r} = \iint dxdy\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$

- $\int_{AB} \bar{F}(\bar{r})d\bar{r}$

  Depend only on begin and end points, not actual contour

  $\int_{C1+C2} \bar{F}d\bar{r} = 0$

  $\int_{C1} \bar{F}d\bar{r} + \int_{C2} \bar{F}d\bar{r} = 0$

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$\bar{F}(x,y,z) = \left(F_x(\bar{r}), F_y(\bar{r}), F_z(\bar{r})\right)$

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$\phi(F) = \phi(x,y,z)$

If $\bar{F}(\bar{r}) = -\nabla\phi$

$F_x = -\frac{\delta\phi}{\delta x}$

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Cross product

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Vector

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$\nabla\times\bar{F} = \left(\frac{\delta}{\delta y}F_z - \frac{\delta}{\delta z}F_y, \frac{\delta}{\delta z}F_x - \frac{\delta}{\delta x}F_z, \frac{\delta}{\delta x}F_y - \frac{\delta}{\delta y}F_x\right)$

$\nabla\phi(\bar{r}) = vector$

$= \left(\frac{\delta\phi}{\delta x}, \frac{\delta\phi}{\delta y}, \frac{\delta\phi}{\delta z}\right)$

Gradient of ϕ

Grad ϕ

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Conservative

$\nabla \times \bar{F} = 0?$

$\nabla \times \bar{F} = (2x^2yz - 2x^2yz - 2xy^2z, 2xy^2z - 2xy^2z, 2xyz^2 + 2 - 2xyz^2 - 2) = (0,0,0)$

$\bar{F}$ is conservative

Ex Vector field $\bar{F}$ follows from potential $\phi(\bar{r}) = \frac{1}{r}$

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$\bar{F} = -\nabla\phi = \left(\frac{\delta\phi}{\delta x}, \frac{\delta\phi}{\delta y}, \frac{\delta\phi}{\delta z}\right)$

$\phi = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$

$\frac{\delta\phi}{\delta x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}2x$

$= -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = -\frac{x}{r^3}$

$\frac{\delta\phi}{\delta y} = -\frac{y}{r^3}$

$\frac{\delta\phi}{\delta y} = -\frac{z}{r^3}$

$\bar{F} = -\nabla\phi$

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$f(x + \Delta x) = f(x) + \Delta x f'(x) + \theta(\Delta x^2)$

$\delta\bar{r} = (\delta x, \delta y, \delta z)$

$$\delta x \frac{\delta\phi}{\delta x} + \delta y \frac{\delta\phi}{\delta y} + \delta z \frac{\delta\phi}{\delta z}$$

$\phi(\bar{r} + \delta\bar{r}) = \phi(\bar{r}) + \delta\bar{r}\nabla\phi + \cdots$

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$= (\nabla \times \bar{F})_z$

$$\oint_C \bar{F} d\bar{r} = \iint_A (\nabla \times \bar{F})_z$$

$$\iint_A (\nabla \times \bar{F})_z$$

Normal vector of surface

$\hat{n}$

$\hat{n}$ is always $\perp$ to surface, $\hat{n}.\hat{n} = 1$

Eg. Surface is in xy-plane $\hat{n} = \hat{z} = (0,0,1)$

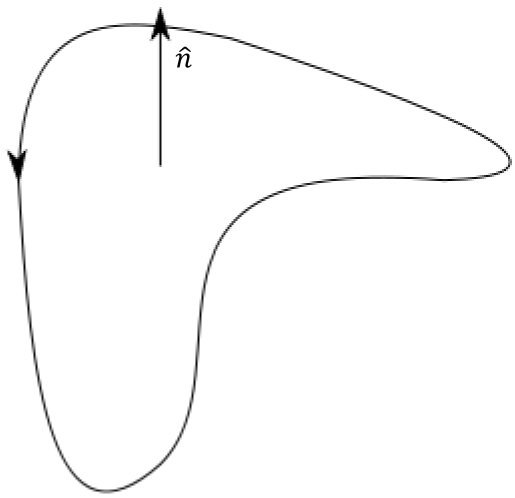
Eg. Sphere: $\hat{n} = \hat{r} = \frac{\bar{r}}{r} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$

$(\nabla \times \bar{F})_z = (\nabla \times \bar{F}) * \hat{n}$

$\hat{n}dxdy = \hat{n}dS = d\bar{S}$

If surface is closed (sphere) then $\hat{n}$ points outwards

If surface is open, i.e. it has a boundary, then the direction of the normal vector follows from Right hand rule



Surface is characterised by

Area

Normal vector

$\iint d\bar{S} = \iint \hat{n} dS$

dS=2D integral

$\oint \bar{F} d\bar{r} = \iint dxdy(\nabla \times \bar{F})_z$

$\hat{n} = (0,0,1)$

Dxdy=dS

$(\nabla \times \bar{F})_z = \hat{n}(\nabla \times \bar{F})$

$= \iint dS\, \hat{n}(\nabla \times \bar{F}) = \iint d\bar{S}(\nabla \times \bar{F})$

3D

$\oint_C \bar{F} d\bar{r} =$

$\iint_A d\bar{S} * (\nabla \times \bar{F})$

Ex HEMISPHERE

$x^2 + y^2 + z^2 = a^2$

$z > 0$

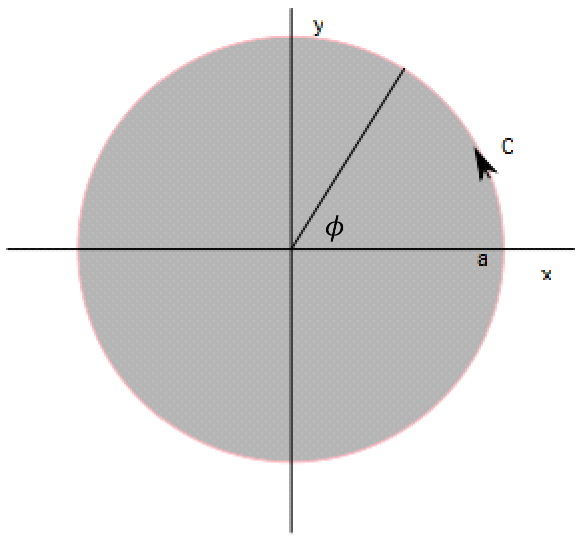
Boundary C

$x^2 + y^2 = a^2, z = 0$

$\bar{F}(\bar{r}) = (-y, x, 0)$

Verify Stokes theorem

1. $\oint_C \bar{F}\, d\bar{r}$



$$\oint_C \bar{F}(\bar{r})d\bar{r} = \int_0^{2\pi} d\phi \frac{d\bar{r}}{d\phi}\bar{F}\big(\bar{r}(\phi)\big) = \int_0^{2\pi} d\phi a^2 = 2\pi a^2$$

$$\frac{d\bar{r}}{d\phi} = jacobian$$

$$\frac{d\bar{r}}{d\phi} = \left(\frac{dx}{d\phi}, \frac{dy}{d\phi}, \frac{dz}{d\phi}\right)$$

$$= (-\mathrm{a}\sin\phi\,, \mathrm{a}\cos\phi\,, 0)$$

$$\bar{F}\big(\bar{r}(\phi)\big) = (-\mathrm{a}\sin\phi\,, \mathrm{a}\cos\phi\,, 0)$$

$$\frac{d\bar{r}}{\mathrm{d}\phi} * \bar{F} = a^2\sin^2\phi + a^2\cos^2\phi + 0 = a^2$$

2. $\iint d\bar{S} * \nabla \times \bar{F}$

$$\hat{n} = \hat{r} = \frac{\bar{r}}{r} = \frac{r}{a}$$

$$= \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)$$

$$\bar{F} = (-y, x, 0)$$

$$\nabla \times \bar{F} = \left(\delta_y\, F_z - \delta_z F_y, \delta_z F_x - \delta_x F_z, \delta_x F_y - \delta_y F_x\right)$$

$$= (0 - 0{,}0 - 0{,}1 - -1)$$

$$= (0{,}0{,}2)$$

$$\iint d\bar{S} * \nabla \times \bar{F} = \iint dS\, \hat{n}\nabla \times \bar{F}$$

$$x = a\cos\phi\sin\theta$$

$$y = a\sin\phi\sin\theta$$

$$z = a\cos\theta$$

$$0 < \phi < 2\pi$$

$$0 < \theta < \frac{\pi}{2}$$

Upper hemisphere

$$\hat{n}\nabla \times \bar{F} = \frac{\bar{r}}{a} * (0{,}0{,}2) = \frac{2z}{a} = \frac{2a\ \cos\theta}{a} = 2\cos\theta$$

$$\text{Jacobian} = r^2\sin\theta = a^2\sin\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} a^2\sin\theta\, 2\cos\theta$$

$$= a^2\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta 2\sin\theta\cos\theta = a^2\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta\sin 2\theta$$

$$= 2\pi a^2\left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2\pi a^2 \left(-\frac{1}{2}\right)(-1-1) = 2\pi a^2$$

Integrals of the type

$\iint \bar{F} d\bar{S}$

Flux of $\bar{F}$ through surface S

$Q = \iint \bar{E} d\bar{S}$

Electromagnetism

$\bar{F} = (0,0,x^2+y^2)$

Plane x=2

Flux=0 since $\bar{F}$//plane $\bar{F} \perp \hat{n}$

Plane z=2

Flux $\neq 0$

$\bar{F} \perp plane$

$\bar{F}//\hat{n}$

Ex

Surface is closed box with side length a

Vector field $\bar{F} = (x,y,z) = \hat{r}$

Consider each side separately

1. Front

$x = a, 0 < y, z < a$

$\hat{n} = (1,0,0)$

$\iint d\bar{S}\bar{F} = \iint dS\, \hat{n} * \bar{F}$

$= \int_0^a dy \int_0^a dz\, F_x(a,y,z)$

$= \int_0^a dy \int_0^a dz\, x \Big|_{x=a}$

$= a^3$

2. Back

$x = 0, 0 < y, z < a$

$\hat{n} = (-1,0,0)$

$\hat{n}\bar{F} = -F_x(0,y,z)$

$= -x \Big|_{x=0} = 0$

3. Top

$z = a, 0 < x, y < a$

$\hat{n} = (0,0,1)$

$\hat{n}\bar{F} = F_z(x,y,a)$

$= a$

$\int_0^a dx \int_0^a dy\, a = a^3$

4. Bottom, z=0

$\hat{n} = (0,0,1)$

$\hat{n}\bar{F} = -F_z(x,y,0)$

$= 0$

5. Left side, y=0

$\hat{n} = (0,-1,0)$

$\hat{n}\bar{F} = -F_y(x,0,z) = 0$

6. Right side

$y = a$

$\hat{n} = (0,1,0)$

$\hat{n}\bar{F} = F_y(x,a,z)$

$\int_0^a dx \int_0^a dz\, F_y(x,a,z)$

$= a^3$

Divergance Theorem

13 December 2011
11:04

(gauss theorem)
Consider a closed surface S in $\mathbb{R}^3$
3D interior volume V
$S = \delta V$
Consider a vector field $\bar{F}(\bar{r})$
$\iint \bar{F}(\bar{r})d\bar{S} = \iiint \nabla * \bar{F}(\bar{r})dV$
Flux through surface = Volume integral of div $\bar{F}$

Proof

Consider a small cube and compute both sides using $\Delta x, \Delta y, \Delta z \ll 1$
We have already computed the flux (see notes)
Flux through left, bottom, back =0, follows from symmetry
Right, front, top: Flux=a^3

$$(a^2 \quad * \quad a)$$
$$Area \quad \bar{F}\Big|_{sides}$$

$\Rightarrow$total flux $3a^3$

Div theorem

$$\nabla.\bar{F} = \frac{\delta}{\delta x}F_x + \frac{\delta}{\delta y}F_y + \frac{\delta}{\delta z}F_z, \bar{F} = \bar{r} = (x, y, z)$$
$$= 1 + 1 + 1 = 3$$

$$\iiint_{cube} \nabla.\bar{F}dV = 3\iiint_{cube} dV = 3a^3$$

Ex sphere, $\bar{F}(\bar{r}) = \bar{r}$

$\iint \bar{F}.d\bar{S}$
$d\bar{S} = \hat{n}dS$
$\hat{n} = \hat{r} = \frac{\bar{r}}{r}$
$dS = a^2 \sin\theta \, d\phi d\theta$

$a^2 \sin\theta = jacobian\Big|_{r=0}$

$$\iint \bar{F}.d\bar{S} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, a^2 \sin\theta \, a$$
$$= a^3 2\pi \int_0^{\pi} d\theta \sin\theta$$
$$= a^3 2\pi[-\cos\theta]_0^{\pi}$$
$$= 4\pi a^3$$

$$\iiint_{sphere} \nabla.\bar{F}dV, \nabla.\bar{F} = 3$$
$$= 3\iiint_{sphere} dV = 3\frac{4\pi}{3}a^3 = 4\pi a^3$$

Ex

$\iint (x^2 + y + z)dS$

S= closed surface of sphere, radius 1

$$x^2 + y^2 + z^2 = r^2 = 1$$

Use Gauss' theorem!
$\iint \bar{F}d\bar{S} = \iiint \nabla\bar{F}dV$
Find $\bar{F}$ such that $\iint \bar{F}d\bar{S} = \iint (x^2 + y + z)dS$

$d\bar{S} = \hat{n}dS, \hat{n} = \hat{r} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$

$\bar{F}.\hat{n} = x^2 + y + z$

$$F_x\frac{x}{r}+F_y\frac{y}{r}+F_z\frac{z}{r}\Big|_{r=1}=x^2+y+z$$

$$F_x x+F_y y+F_z z=x^2+y+z$$

$$\rightarrow \begin{matrix} F_x=x \\ F_y=F_z=1 \end{matrix}$$

$$\bar{F}(\bar{r})=(x,1,1)$$

Now RHS

$$\nabla.\bar{F}=\frac{\delta}{\delta x}x+\frac{\delta}{\delta y}1+\frac{\delta}{\delta z}1=1$$

$$\iiint_{\text{sphere}}\nabla.\bar{F}dV=\iiint_{sphere}dV=\frac{4\pi}{3}$$

<u>Recap of integral theorems</u>

Ordinary integration

$$I=\int_a^b dx\, f(x)\, , f(x)=\frac{dg(x)}{dx}$$

g(x)= primative

$$[=g(x)\Big|_a^b=g(b)-g(a)$$

-end points

"surface terms" boundary

Conservative vector field $\bar{F}(\bar{r})$

$\Leftrightarrow \nabla\times\bar{F}=\bar{0}$

$\Leftrightarrow \bar{F}=-\nabla\phi$

ϕ is a potential

$$I=\int_a^b\bar{F}d\bar{r}=-\int_a^b d\bar{r}\,\nabla\phi$$

$$=-\big(\phi(B)-\phi(A)\big)$$

"surface term" boundaries

Work done= difference in potential energy

Stokes' theorem

2D:

$$\iint_A\left(\frac{\delta}{\delta x}F_y-\frac{\delta}{\delta y}F_x\right)dxdy=\oint_c\bar{F}(\bar{r})d\bar{r}$$

3D:

$$\iint_A(\nabla\times\bar{F}).d\bar{S}=\oint_c\bar{F}d\bar{r}$$

A= area, C=boundary "surface term", $\nabla\times\bar{F}$="derivative"

Divergence theorem

$$\iiint_V\nabla.\bar{F}dV=\iint_S\bar{F}d\bar{S}$$

Boundary

$S=\delta V$

REVISION CLASS Fri 13/1 Glyn E

14 December 2011
11:05

1) $\int_{\bar{\sigma}(\phi)} \bar{F}(\bar{r}).d\bar{r}$

$\bar{F}(\bar{r}) = (-xy, x^2)$
$\bar{\sigma}(\phi) = (\cos\phi, \sin\phi)$
$0 < \phi < \frac{\pi}{2}$
$x = \cos\phi$
$y = \sin\phi$
$\int \bar{F}(\bar{r})d\bar{r} = \int_0^{\frac{\pi}{2}} d\phi \frac{d\bar{\sigma}}{d\phi} \bar{F}\big(\bar{\sigma}(\phi)\big)$

1. $\frac{d\bar{\sigma}}{d\phi} = (-\sin\phi, \cos\phi)$
   $x = \sigma_x(\phi) = \cos\phi$
   $y = \sigma_y(\phi) = \sin\phi$
2. $\bar{F}\big(\bar{\sigma}(\phi)\big) = (-\cos\phi \sin\phi, \cos^2\phi)$
3. $\frac{d\bar{\sigma}}{d\phi}\bar{F} = c\phi s^2\phi + c^2\phi$
   $c\phi(s^2\phi c^2\phi) = c\,\phi$
4. $\int_0^{\frac{\pi}{2}} d\phi c\phi = s\phi \Big|_0^{\frac{\pi}{2}} = 1$

2) $\bar{A}(\bar{r}) = (xyz^2 + x^2, y + xz, z^2)$

1. $\bar{A}.\bar{A}$ scalar
   $= A_x^2 + A_y^2 + A_z^2$
2. $\nabla.\bar{A} = div\ \bar{A}$
   $\nabla = nabla$
   $= \left(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}\right)$
   $\nabla.\bar{A} = \frac{\delta}{\delta x}A_x + \frac{\delta}{\delta y}A_y + \frac{\delta}{\delta z}A_z$
   $= yz^2 + 2x + 1 + 2z$
3. $\nabla(\nabla.\bar{A})$ $grad\ div\ \bar{A}$
   $\nabla.\bar{A} = \phi$
   $\nabla\phi\ vector$
   $= \left(\frac{\delta}{\delta x}\phi, \frac{\delta}{\delta y}\phi, \frac{\delta}{\delta z}\phi\right)$
   $= (2, z^2, 2yz + 2)$
4. $\nabla^2\bar{A}\ vector$
   $\nabla^2 = \nabla * \nabla\ scalar$
   $= \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$
   $= \left(\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z\right)$
   $= (2 + 0 + 2xy, 0 + 0 + 0, 0 + 0 + 2)$

   $\nabla(\nabla.\bar{A}) = \bar{w}$
   $\nabla_j \nabla_i A_i = w_j$

   $\nabla^2\bar{A} = \bar{w}$
   $\nabla_i \nabla_i A_j = w_j$
5. $\nabla \times \bar{A} =$
   $\left(\delta_y A_z - \delta_z A_y, \delta_z A_x - \delta_x A_z, \delta_x A_y - \delta_y A_x\right)$

6. $\nabla * \nabla \times \bar{A} = \nabla * \bar{B}$

Scalar

$= -1 + 2xz + 1 - 2xz$

=0

Not surprise

$\bar{a} \times \bar{b} = \bar{c}$

$\bar{a}.\bar{b} \times \bar{c} = \bar{c}.\bar{a} \times \bar{b} = \bar{b}.\bar{c} \times \bar{a}$

Cyclic

$\bar{a}.\bar{a} \times \bar{b} = \bar{b}.\bar{a} \times \bar{a}$

$\bar{a} \times \bar{b} = \bar{c} \perp \bar{a}$

$\bar{a}.\bar{c} = 0$

$\bar{A}.\ \nabla\ \bar{A} = \bar{H}$

$\bar{A} * \nabla = \mathrm{A_x}\frac{\delta}{\delta\mathrm{x}} + \mathrm{A_y}\frac{\delta}{\delta y} + A_z\frac{\delta}{\delta z}$

$\neq \bar{A}\nabla.\bar{A}$

$\nabla.\bar{A} = div\ \bar{A}$

$\bar{A} * \nabla = \bar{H} = \left(\bar{A}\nabla\mathrm{A_x}, \bar{A}\nabla\mathrm{A_y}, \bar{A}\nabla\mathrm{A_z}\right)$

3) $\bar{F}(\bar{r}) = (3x^2y^2z, 2x^3yz, x^3y^2)$

$\frac{d\bar{\sigma}}{dt} = (1{,}2t, 3t^2)$

$\bar{F}\left(\bar{\sigma}(t)\right)$

$x = t$

$y = t^2$

$z = t^3$

1. Conservative

   $\nabla \times \bar{F} = (0{,}0{,}0)$

2. $\phi\ scalar, 2\ function$

   $\bar{F} = -\nabla\phi$

   $F_x = -\frac{\delta}{\delta x}\phi = x^2y^2z\ (1)$

   $F_y = -\frac{\delta}{\delta y}\phi = 2x^3yz\ (2)$

   $F_z = -\frac{\delta}{\delta z}\phi = x^3y^2\ (3)$

   (1) $\frac{\delta\phi}{\delta x} = -3x^2y^2z$

   (2) $\frac{\delta\phi}{\delta y} = -2x^3yz$

   (3) $\frac{\delta\phi}{\delta z} = -x^3y^2$

   $\phi(x, y, z) = -x^3y^2z + f(y, z)$

   $\phi(x, y, z) = -x^3y^2z + g(x, z)$

   $\phi(x, y, z) = -x^3y^2z + h(x, y)$

   $\phi(x, y, z) = x^3y^2z + c$

3. $\int_{\bar{\sigma}(t)} \bar{F}(\bar{r})d\bar{r}$

   0<t<1

   $\bar{\sigma}(t) = (t, t^2, t^3)$

   $\int_0^1 dt\ \frac{d\bar{\sigma}}{dt}.\bar{F}\left(\bar{\sigma}(t)\right)$

   $F\left(\bar{\sigma}(t)\right) = (3t^9, 2t^8, t^7)$

   $\frac{d\bar{\sigma}}{dt}.\bar{F} = 3t^9 + 4t^9 + 3t^9 = 10t^9$

$$\int_0^1 dt\ 10t^9 = t^{10}\Big|_0^1 = 1$$

4) $\bar{G}(\bar{r}) = (x^2y, xy)$

1. $y = x$ $(0,0) \to (1,1)$

   $\bar{\sigma}(t) = (t, t)$

   $0 < t < 1$

   $\int d\bar{r}\bar{G}(\bar{r}) =$

Hamiltonian systems

04 October 2011
13:02

1. Phase space Γ
2. Hamiltonian $H: \Gamma \to \mathbb{R}$
3. Equations of motion $\begin{cases} \dot{q} = \frac{\delta H}{\delta p} \\ \dot{p} = -\frac{\delta H}{\delta q} \end{cases}$
4. $Boundary\ conditions$ $\begin{cases} q(0) = q_0 \\ p(0) = p_0 \end{cases}$

Example 2: harmonic oscillator

- Lagrangian

$$L = \frac{1}{2}m\,\dot{q}^2 - \frac{1}{2}m\omega^2 q^2$$

$$E.o.m.:\ 0 = \frac{\delta}{\delta q}L - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right)$$

$$= -mw^2 q - m\ddot{q}$$

$$\boxed{\ddot{q} = -\omega^2 q}$$

- Newtonian mechanics

$$F = ma$$

$$F = -kq$$

$$m\ddot{q} = -kq$$

$$\Rightarrow \ddot{q} = -\omega^2 q$$

$$\omega^2 \equiv \frac{k}{m}$$

- Hamiltonian case

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$$

(total energy!)

$$e.o.m.:\ \ \dot{q} = \frac{\delta H}{\delta p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\delta H}{\delta q} = -m\omega^2 q$$

Time derivative of first

$$\ddot{q} = \frac{\dot{p}}{m} = -\omega^2 q$$

Legendre transform

Formal technique, allowing to reformulate problems in equivalent ways

Def: start from lagrangian eg $L = \frac{1}{2}m\dot{q}^2 - V(q)\quad L(q, \dot{q})$

Define momentum $p \equiv \frac{\delta L}{\delta \dot{q}} = m\dot{q}$

Define $H = \dot{q}p - L$

$$= \frac{1}{2m}p^2 + V(q)$$

Thermodynamics

06 October 2011
13:22

Thermodynamic system: any macroscopic system
Thermodynamic parameters : V,T,P,N,… things that can be measured w/o disrupting system
Thermodynamic state: specify parameters
Thermodynamic equilibrium: no time dependence
Equation of state: functional relation among parameters
Thermodynamic transformation: change of state
    Quasi-static: intermediate steps equilibrium
    Reversible: quasi static, goes through same steps on way back
Work W: from mechanics
Heat Q: thing exchanged when temperature changes without producing work
Heat capacity: $C = \frac{\Delta Q}{\Delta T}$
Heat reservoir: very big system, such that $C \to \infty$
Ideal gas: limiting case of a diluted gas
    N <u>identical</u> particles
    Point-like particles
    Interactions short range

<u>State function</u>
Any function which depends on the state occupied by the system (parameters) but <u>not</u> on the history of the system

1. U(internal energy), S(entropy) are state functions
2. $\int_\gamma du = o = \int_\gamma ds$
   Where γ is a closed path in the space of parameters
3. $dU \text{ } and \text{ } dS$ are exact differentials (it is <u>NOT</u> true for W, Q)
4. U, S defined (<u>classically</u>) up to a constant

Thermodynamic potential
Helmoltz free energy
$F \equiv U - TS$
Gibbs free energy
$G \equiv F + PV$
Hentalpy
$H \equiv U + PV$

Digression: differentiating functions of many variables

12 October 2011
09:07
$f: \mathbb{R}^m \to \mathbb{R}$
Differential
$$df = \left(\frac{\delta f}{\delta x_1}\right) dx_1 + \left(\frac{\delta f}{\delta x_2}\right) dx_2 + \cdots$$
Partial derivatives
$$\frac{\delta f}{\delta x_i}$$
$f(x_1 + x_2 + \cdots + x_m)$
"vector"
$$\left(\frac{\delta f}{\delta x_1}, \frac{\delta f}{\delta x_2}, \dots\right)$$
Second differential is m*m matrix
It is a <u>symmetric</u> matrix

$$\frac{\delta^2 f}{\delta x_i \delta x_j} = \frac{\delta^2 f}{\delta x_j \delta x_i}$$

1st law thermodynamics

The quantity $\boxed{dU \equiv \delta Q - \delta W}$ (conventional!)
Is an exact differential and it defines the state function U (internal energy)

2nd law thermodynamics

The quantity

$$dS \equiv \frac{\delta Q}{T}$$

Is exact diff. for reversible infinitesimal transformations and defines a state function S (entropy)
Also: the entropy of a thermally isolated system is non-decreasing (clausius theorem)

3rd law thermodynamics

The entropy S at $T \rightarrow 0$ is universal (=does not depend on the system) and can be chosen to vanish
$\boxed{S(T \rightarrow 0) = 0}$

Exercise: internal energy of ideal gas

1. dU and dS exact
2. Eq. of state $PV = NkT$

Show that $U = U(t)$
Depends only on T

$$U = \frac{3}{2} NkT$$

Proof:

1st law: $\delta Q = dU + \delta W$
$\delta W = P\,dV$
2nd law: $\delta Q = T\,ds$

$$\delta Q = \frac{\delta U}{\delta T} + \frac{\delta U}{\delta V} dV + P\,dv$$

Assuming that U=U(V,T)

$$dS = \frac{1}{T}\left(\frac{\delta U}{\delta T}\right) dT + \frac{1}{T}\left(\frac{\delta U}{\delta V} + P\right) dV$$

$$\frac{\delta}{\delta V}\left[\frac{1}{T}\frac{\delta U}{\delta T}\right] = \frac{\delta}{\delta T}\left[\frac{1}{T}\left(\frac{\delta U}{\delta V} + P\right)\right]$$

From the fact that the matrix of second derivatives is symmetric

"who got lost?"
Most of class raises hand
"very good"

$$\boxed{\begin{gathered} if\ U = U(V,T), \\ by\ definition \\ dU = \frac{\delta U}{\delta V} dV + \frac{\delta U}{\delta T} dT \end{gathered}}$$

$$\frac{1}{T}\left(\frac{\delta U}{\delta T}\right) dT \Rightarrow \frac{\delta S}{\delta T}$$

$$\frac{1}{T}\left(\frac{\delta U}{\delta V} + P\right) dV \Rightarrow \frac{\delta S}{\delta V}$$

$$\frac{\delta}{\delta V}\left[\frac{1}{T}\frac{\delta U}{\delta T}\right] \Rightarrow \frac{\delta}{\delta V}\frac{\delta}{\delta T} S$$

$$\frac{\delta}{\delta T}\left[\frac{1}{T}\left(\frac{\delta U}{\delta V} + P\right)\right] \Rightarrow \frac{\delta}{\delta T}\frac{\delta}{\delta V} S$$

$$\frac{1}{T}\frac{\delta^2}{\delta V \delta T} U = \frac{1}{T}\frac{\delta^2}{\delta T \delta V} - \frac{1}{T^2}\left(\frac{\delta U}{\delta V} + P\right) + \frac{1}{T}\frac{\delta}{\delta T} P$$

$$dU \; exact \Rightarrow \frac{1}{T}\frac{\delta^2}{\delta V \delta T}U = \frac{1}{T}\frac{\delta^2}{\delta T \delta V}$$
$$0 = -\frac{1}{T^2}\frac{\delta U}{\delta V} - \frac{P}{T^2} + \frac{1}{T}\frac{\delta P}{\delta T}$$
$$PV = NkT$$
$$\boxed{P = \frac{NkT}{V}}$$

$$\frac{\delta P}{\delta T} = \frac{\delta}{\delta T}\left(\frac{NkT}{V}\right) = \frac{Nk}{V}$$
$$T\frac{\delta P}{\delta T} = \frac{NkT}{V} = P$$

$$0 = \frac{1}{T^2}\left(-\frac{\delta U}{\delta V} - P + T\frac{\delta P}{\delta T}\right)$$
$$0 = \frac{1}{T^2}\left(-\frac{\delta U}{\delta V} - P + P\right) \Rightarrow 0 = \frac{1}{T^2}\left(-\frac{\delta U}{\delta V}\right)$$
$$T \neq 0 \Rightarrow \boxed{\frac{\delta U}{\delta V} = 0}$$

Maxwell Relations

Set 1: $dU = \delta Q - \delta W$ (1st law)

$= TdS - PdV$ (2nd law)

$$T = \left(\frac{\delta U}{\delta S}\right)_V$$
$$P = -\left(\frac{\delta U}{\delta S}\right)_S$$

It is <u>natural</u> to write U as a function U(S,V) of S,V

δQ: infinitesimal amount of heat given to a system

δW infinitesimal amount of work done by a system

Set 2: Helmalts Free Energy

$F \equiv U - TS$

Differentiate

$dF = dU - d(TS)$

$= dU - SdT - TdS$

$= TdS - PdV - SdT - TdS$

$\boxed{dF = -PdV - SdT}$

$\Rightarrow F = F(T,V)$

$$S = -\left(\frac{\delta F}{\delta T}\right)_V$$
$$P = -\left(\frac{\delta F}{\delta V}\right)_T$$

Set 3 Gibbs Free Energy

$G = F + PV$

$dG = -SdT + VdP$

$$S = -\left(\frac{\delta G}{\delta T}\right)_P$$
$$V = \left(\frac{\delta G}{\delta P}\right)_T$$

Set 4 Enthalpy

$H = G + TS$

$dH = TdS + VdP$

$$T = \left(\frac{\delta H}{\delta S}\right)_P$$
$$V = \left(\frac{\delta H}{\delta P}\right)_S$$

$U(S,V) \quad \leftrightarrow_{S-T} \quad F(T,V)$

$\updownarrow_{P-V} \qquad\qquad \updownarrow_{P-V}$

$H(S,P) \quad \leftrightarrow_{S-T} \quad G(T,P)$

Legendre Transform!
Analogous to

$$L = L(q, \dot{q}) \overset{\dot{q}-p}{\longleftrightarrow} H = H(q, p)$$

Classical Statistical Mechanics

13 October 2011
09:28

- We study systems
  - Large number particles N
  - Occupy volume V (assumed to be finite)
  - Equilibrium!
- Basic Strategy
  - Replace/generalize the method of "most likely distribution" by sets of counting exercises and "ensemble averages"
- Input
  - Phase-space Γ
  - Hamiltonian
  - Density function
    $\rho(q,p)$
    - Counts number of microscopic states as a function of phase-space variables
  - All physical quantities (macroscopic level) are computed as ensemble averages:

$$< f > \equiv \frac{\int_\Gamma d^{3N}q d^{3N}p \rho(q_i,p_j) f(q_i,p_j)}{\int_\Gamma d^{3N}q d^{3N}p\rho(q_i,p_j)}$$

$$f \Rightarrow any\ function$$

$\boxed{microcanonical\ ensemble}$

Consider system isolated

1. N is fixed
2. E=U is fixed

   This defines ρ to be

$$\rho = \begin{cases} 1\ if\ E < H < E + \Delta \\ 0\ otherwise \end{cases}$$

Where Δ infintesimal

Microcanonical Ensemble

Assume system isolated

$$\begin{cases} U = E & Fixed \\ N & Fixed \end{cases}$$

Δ small energy interval ($\Delta \ll E$)

$$\begin{cases} 1 & if & E < H < E + \Delta \\ 0 & otherwise \end{cases}$$

$$\sum(E) = \int_{\Gamma \atop H<E}^{0} d^{3N}q d^{3N}p$$

Volume of phase space with E>H

$$d^{3N}q \equiv dq_1 dq_2 dq_3 \dots dq_{3n}$$

$$\Omega(E) \equiv \int_\Gamma d^{3n}q d^{3N}p\rho(p,q)$$

Volume of phase space occupied by the ensemble

$$= \int_\Gamma d^{3N}q d^{3n}p$$

$$E < H < E + \Delta$$

$$= \sum(E+\Delta) - \sum(E)$$

$$\omega(E) \equiv \frac{\delta \sum E}{\delta E} \Rightarrow \Omega(E) = \Delta\omega(E)$$

$$\Delta \ll E$$

$\boxed{S(E,V,N) \equiv \kappa \ln \Omega(E)}$

We identify S with entropy

In thermodynamics

$$dS = \frac{\delta Q}{T}$$

1. S is Estensive

    (the S if a system is the sum of S for subsystems)

$N = N_1 + N_2$
$V = V_1 + V_2$
$H = H_1 + H_2$
↑ CAREFUL! Assumes short-range interactions
$S(E_1, V_1, N_1) = k \ln \Omega_1(E_1)$
$S(E_2, V_2, N_2) = k \ln \Omega_2(E_2)$
$\Omega(E) = \int d^{3N}q d^{3N}p$

$E < H < E + \Delta$

$$= \boxed{\sum_{m=0}^{\frac{E}{\Delta}} \int_{E_1<H_1<E_1+\Delta} d^{3N_1}q_1 d^{3N_1}p_1 \int_{E_2<H_2<E_2+\Delta} d^{3N_2}q_2 d^{3N_2}p_2 \,, E = E_1 + E_2}$$

$$= \sum_{m=0}^{\frac{E}{\Delta}} \Omega_1(E_m) \times \Omega_2(E - E_m)$$

$E_m = m\Delta$

Notice: all terms in the sum are positive and finite.

CAPS LOCK IS CRUISE CONTROL FOR COOL!

$\exists \bar{E}$ such that

$\Omega_1(\bar{E})\Omega_2(E - \bar{E}) \geq \Omega_1(E_m)\Omega_2(E - E_m)$

$\forall m$

$$\Omega_1(\bar{E})\Omega_2(E - \bar{E}) \leq \Omega(E) \leq \frac{E}{\Delta}\Omega_1(\bar{E})\Omega_2(E - \bar{E})$$

$\frac{E}{\Delta} = total\ number\ of\ terms$

Take Logarithm

$$S_1(\bar{E}) + S_2(E - \bar{E}) \leq S(E) \leq S_1(\bar{E}) + S_2(\bar{E}) + k \ln\left(\frac{E}{\Delta}\right)$$

Take N>>1

1. E is estensive $\Rightarrow E \propto N$

$vol(6N\ dimensional\ space) \sim x^{6N}$

$$\lim_{N\to\infty}\left(\frac{\ln\frac{E}{\Delta}}{S_1(\bar{E}) + S_2(E - \bar{E})}\right) = \lim_{N\to\infty}\frac{\ln N}{6N} \neq 0$$

$\Rightarrow S(E) = S_1(\bar{E}) + S_2(E - \bar{E})$

AT LARGE N

1. Entropy S is estensive
2. S can be used to <u>define</u> temperature T

    Proof: we already showed that there exists $\bar{E}$ such that

    $S(E) = S_1(E) + S_2(E - \bar{E})$

    $\Omega(E) = \Omega_1(\bar{\mathrm{E}})\Omega_2(\mathrm{E} - \bar{\mathrm{E}})$

    $\bar{E}$ yields the maximum contribution

$$\begin{cases} d(\Omega_1(\mathrm{E}_1)\Omega_2(\mathrm{E}_2))\Big|_{\mathrm{E}_1=\bar{\mathrm{E}}} = 0 \\ d(E_1 + E_2) = 0 \end{cases}$$

    Take log

$$\begin{cases} d(\ln \Omega_1(\mathrm{E}_1)\Omega_2(\mathrm{E}_2))\Big|_{\mathrm{E}_1=\bar{\mathrm{E}}} = 0 \\ d(E_1 + E_2) = 0 \end{cases}$$

$d\ln(\ln \Omega_1(\mathrm{E}_1)\Omega_2(\mathrm{E}_2))$

$$= dE_1 \frac{\delta \ln \Omega_1}{\delta E_1} + dE_2 \frac{\delta \ln \Omega_2}{\delta E_2}$$

$= dE_1 \left(\frac{\delta \ln \Omega_1}{\delta E_1} - \frac{\delta \ln \Omega_2}{\delta E_2} \right)$

Multiply by κ

$\frac{\delta S_1}{\delta E_1}\Big|_{E_1=\bar{E}} = \frac{\delta S_2}{\delta E_2}\Big|_{E_2=E-\bar{E}}$

Hence define

$\frac{1}{T} \equiv \frac{\delta S}{\delta E}$

3. Equivalent definitions

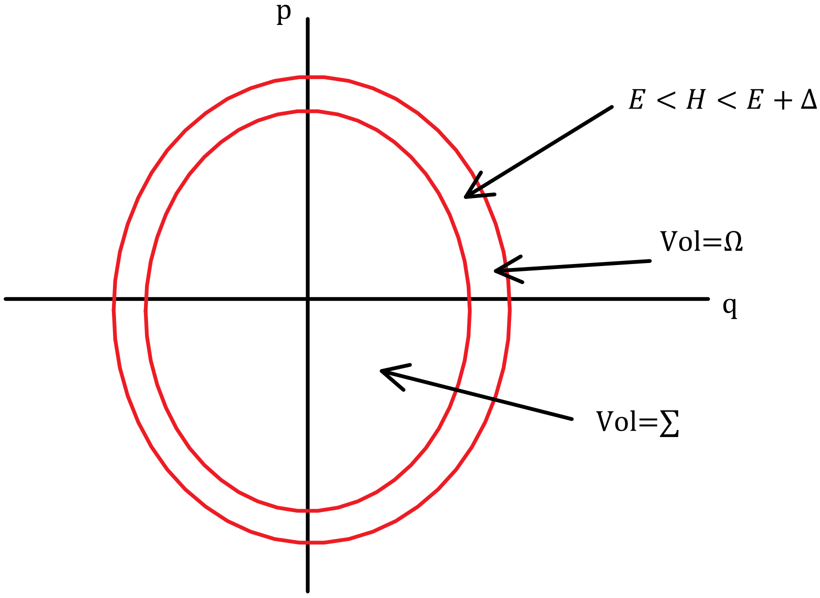
   $S(E,V,N) \equiv \kappa \ln \sum(E)$

   $S(E,V,N) \equiv \kappa \ln \omega(E)$

   "equivalent" means: different, but yields same thermodynamics when we take $N \to \infty$

   $S = \kappa \ln \Omega$

   Example



4. 2nd Law

   "S is a non-decreasing function"

   Proof

   $S = S(E,N,V)$

   But E,N fixed

   Only V can change

   V can only grow

   $\Sigma = \int_{H<E} d^{3N}q d^{3N}p$

   If V grows, the phase-space is growing

   $\Sigma \; grows \Rightarrow S \; grows$

5. 1st Law

   Proof

   $S = S(E,N,V)$

   Keep N fixed and differentiate

   $dS = dE \frac{\delta S}{\delta E} + dV \frac{\delta S}{\delta V}$

   Use def of T

   $= \frac{1}{T} dE + dV \frac{\delta S}{\delta V}$

   $\Rightarrow dE = Tds - \left(T \frac{\delta S}{\delta V} \right) dV$

   Define

   $P \equiv T \frac{\delta S}{\delta V}$

   $\boxed{dE = T\,dS - P\,dV}$

Recap

Define

$$S = \kappa \ln \Omega$$

S has following propeties

1. Estensive
2. Defines T (equilibrium)
3. Equivalent definitions
4. 2nd law
5. 1st law

General Prescription

1. Compute Σ (or Ω, ω)
2. Derive $S = \kappa \ln \Sigma$

   And take $N \to \infty$ (very large)
3. Invert relation

   $S(E) \to E(S)$
4. Reconstruct thermodynamics

$$T = \frac{\delta U}{\delta S}$$

$$P = -\frac{\delta U}{\delta V}$$

$$F = U - TS$$

$$C_V = \frac{\delta U}{\delta T}$$

5. Derive equation of state

`````

Sphere

$$S^{D-1} \quad \sum_{i=1}^{D} q_i^2 = R^2$$

$$\begin{cases} vol(S^{D-1}) = \dfrac{2\pi^{\frac{D}{2}}}{\Gamma_E\left(\frac{D}{2}\right)} \\ vol\ (B^D) = \dfrac{\pi^{\frac{D}{2}} R^D}{\Gamma_E\left(1 + \frac{D}{2}\right)} \end{cases}$$

$\Gamma_E$ is euler gamma function

def

$$\Gamma_E(z) = \int_0^\infty dt\ t^{z-1} e^{-t}$$

properties

$$\begin{cases} x\Gamma_E(x) = \Gamma_E(1+x) \\ \Gamma_E(1+m) = m! \quad m \in \mathbb{N} \\ \Gamma_E\left(\frac{1}{2}\right) = \sqrt{\pi} \end{cases}$$

Examples

D=2

$$vol(S') = \frac{2\pi}{\Gamma(1)} = 2\pi$$

D=3

$$vol(S') =$$

`````````````````````

Ideal gas

Def

N (large!) identical, classical, free, point like particles of mass m in volume V

Phase space

$$\Gamma = \{(q_i, p_j\}, i, j = 1 \dots 3N, p_i, q_i \in \mathbb{R}\}$$

Hamiltonian

$$H = \frac{1}{2m}\sum_{i-1}^{3N} p_i^2$$

Exercise: use its canonical ensemble to

1. Comute

$$\Sigma(E) = \int_\Gamma d^{3N}q d^{3N}p$$

H < E

$$= \int d^{3N}q \int_{H<E} d^{3N}p$$

$$= V^N \int_{H<E} d^{3N}p$$

$$V \equiv \int d^{3N}q$$

$$H = \frac{1}{2m}\sum_i p_i^2 < E$$

$$\sum_{i=1}^{3N} p_i^2 < 2mE$$

The integration is restricted to a 3N dimensional ball B^{3N} in momentum space

With a radius

$$R = \sqrt{2mE}$$

Hence

$$\boxed{\Sigma(E) = V^N \frac{\pi^{\frac{3N}{2}} R^{3N}}{\Gamma_E\left(1 + \frac{3}{2}N\right)}}$$

$$= \frac{\left(2\pi m E V^{\frac{2}{3}}\right)^{\frac{3N}{2}}}{\left(\frac{3N}{2}\right)!}$$

(N even!)

2. $S = \kappa \ln \Sigma$

$$= \kappa \ln \frac{\left(2\pi m E V^{\frac{2}{3}}\right)^{\frac{3N}{2}}}{\left(\frac{3N}{2}\right)!}$$

Take N→ large

Stirling approximation

$\ln m! \approx m \ln m - m$

$$= \frac{3}{2}N\kappa \ln\left(\frac{\left(2\pi m E V^{\frac{2}{3}}\right)}{\frac{3}{2}N\kappa \ln\frac{3}{2}N + \frac{3}{2}N\kappa}\right)$$

$$= \ln e$$

$$= \frac{3}{2}N\kappa \ln\left[\frac{3\pi e}{3} m\ e = EV^{\frac{2}{3}}\right]$$

Define

$$\mu_0 \equiv \frac{4\pi e}{3} M$$

$$\boxed{S = \frac{3}{2}N\kappa \ln\left(\frac{\mu_0 E V^{\frac{2}{3}}}{N}\right)}$$

3. Invert $S(E) \to E(S)$

$$\frac{2S}{3N\kappa} = \ln\left(\frac{\mu_0 E V^{\frac{2}{3}}}{N}\right)$$

$$\exp\left[\frac{2S}{3N\kappa}\right] = \frac{\mu_0 E V^{\frac{2}{3}}}{N}$$

$$\boxed{E = \frac{N}{\mu_0 V^{\frac{2}{3}}} \exp\left[\frac{2S}{3N\kappa}\right]}$$

4. Thermodynamics

$$T = \frac{\delta E}{\delta S}$$

Maxwell relation

$$= \frac{2}{3N\kappa} E \Rightarrow \boxed{E = \frac{3}{2} N\kappa T}$$

$$\Rightarrow C_V = \frac{\delta E}{\delta T} = \frac{3}{2} N\kappa$$

$$P = -\frac{\delta E}{\delta V}$$

Maxwell relation

$$= -\frac{\left(-\frac{3}{2}\right)1}{V} E = \frac{2E}{3V}$$

$$= \frac{2}{3V}\left(\frac{3}{2} N\kappa T\right)$$

$$\boxed{P = \frac{N\kappa T}{V}}$$

2. Gibbs paradox

Ideal gas (example)

$$S = \frac{3}{2} Nk \ln \frac{\mu_0' E v^{\frac{2}{3}}}{N}$$

From explicit caclulation

$$E = \frac{3}{2} NkT$$

$$\boxed{S = \frac{3}{2} Nk \ln cTV^{\frac{2}{3}}}$$

Compute difference in entropy

$$\Delta S = S - S_1 - S_2$$

$$= \frac{3}{2} Nk \ln\left(cTV^{\frac{2}{3}}\right) - \frac{3}{2} N_1 k \ln\left(cTV_1^{\frac{2}{3}}\right) - \frac{3}{2} N_3 k \ln\left(cTV_3^{\frac{2}{3}}\right)$$

$$\Delta S = k\left(N_1 \ln \frac{V}{V_1} + N_2 \ln \frac{V}{V_2}\right)$$

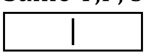
$$V > U_1, U_2$$

$$\Rightarrow \boxed{\Delta S > 0}$$

Compare to

1 $\boxed{blue}$ 2 $\boxed{yellow}$

Same T,P, suppose the 2 gasses have different "colour"



Solution to the paradox

New rule:
Any time there are N <u>indistinguishable</u> particles, add a factor of 1/N!

$$\Sigma(E) = \frac{1}{p^{3N} N!} \int_{H<E} d^{3N}q d^{3N}p$$

$S_{new} = S_{old} - k \ln N!$

N>>1

$= S_{0ld} - kN \ln N$

$= \frac{3}{2} Nk \ln c\, TV^{\frac{2}{3}} - kN \ln N$

$$\boxed{S = \frac{3}{2} Nk \ln C\, T \left(\frac{V}{N}\right)^{\frac{2}{3}}}$$

$$\Delta S = k\left(N\, kn \left(\frac{V}{N}\right)^{\frac{2}{3}} - N_1 \ln \left(\frac{V_1}{N_1}\right)^{\frac{2}{3}} - N_2 \ln \left(\frac{V_2}{N_2}\right)^{\frac{2}{3}}\right.$$

$= 0$

$$Because\ \frac{V}{N} = \frac{V_1}{N_1} = \frac{V_2}{N_2}$$

In the case of different gases

$$\Sigma = \frac{1}{p^{3N}} \frac{1}{N_1! N_2!} \int_{H<E} d^{3N}q\, d^{3N}p$$

$$\frac{1}{N!} \neq \frac{1}{N_1!} \frac{1}{N_2!} \Rightarrow \boxed{\Delta S > 0}$$

Still true for different gases

3. Third Law?

$$S = \frac{3}{2} Nk \ln C\, T \left(\frac{V}{N}\right)^{\frac{2}{3}}$$

When $T \to 0$ problem

Keep c, V, N constant

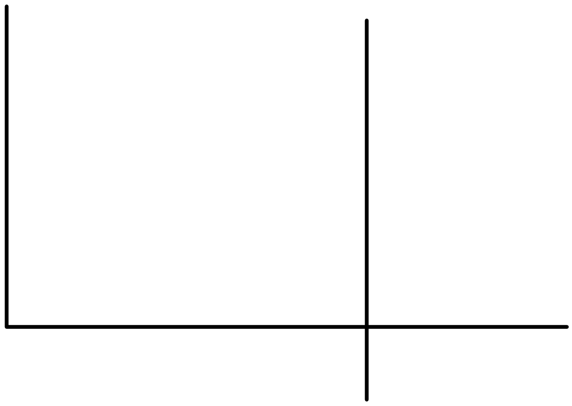
$\lim_{T \to 0} S \to -\infty$

Exercise: paramagnet

1. N particles
2. E energy (fixed)
3. $\Gamma = \{\{\sigma_i\}, \sigma_i = \pm 1\}$
4. $H = -\mu h_M \sum_{i=1}^{N} \sigma_i$

$h_M \Rightarrow$magnetic field

Compute entropy



Exercise: 2 state system (model of paramagnet)
Using microcanonical ensemble (in qm)
Microscopic description

1. N particles (fixed)
2. E energy (fixed)
3. $\Gamma = \left\{ \{\sigma_i\}_?^N, \sigma_i = \pm 1 \right\}$
   Ie requires ?
4. $H = -\mu p_m \Sigma_{i=1}^N \sigma_i$
   ?????

No kinetic or potential term in H

Macroscopic description
N_+
  # occurrences of '+'
N_-
  #occurrences of '-'

This is a QM system!
  Discrete set of states
$$\Omega \equiv \frac{1}{h^{3N}} \int_{H<E} d^{3N}q d^{3N}p \xrightarrow{REPLACE} \Omega: \#\, microstates\ (\{+ \cdots \pm\})$$
  Satisfying macroscopic constraints counting!

Two constraints: E,N fixed
$$\begin{cases} E = -\mu P_m (N_+ - N_-) \\ N = N_1?????? \end{cases}$$

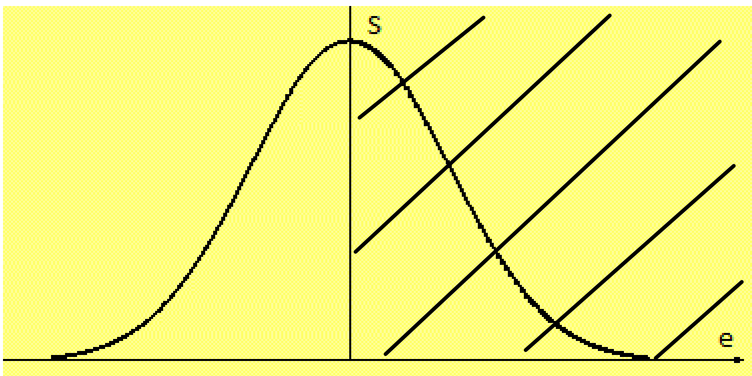
$$\epsilon \equiv \frac{E}{\mu P_m N}$$
$$N_\pm \equiv \frac{N}{2}(1 \mp \epsilon)$$
$$-1 \le \epsilon \le +1$$
$$\Omega = \binom{N}{N_+}$$
$$= \frac{N!}{N_+!\,(N - N_+)!} = \frac{N!}{N_+!\,N_-!}$$

N_+ *and* N_-
known. You want to know how many different sequences $\{+ - + + \ldots +\}$ contain exactly
N_+ $+$
N_- $-$

Start with one such sequence
$\{+,+,\ldots+,-,-,-,\ldots,-\}$
All others obtaineed by changing order
$\Rightarrow factor\ of\ N!$
BUT: overcounting! Exchanging identical symbols yields same sequence
$\Rightarrow factors\ of \frac{1}{N_+!\,N_-!} \Rightarrow \Omega = \frac{N!}{N_+!\,N_-!}$

$S = k \ln \Omega$
$= k \ln \frac{N!}{N_+!\,N_-!}$
Stirling approximation
$\ln N! \sim N \ln N - N$
$= k(N \ln N - N - N_+ \ln N_+ + N_+ - N_- \ln N_- + N_-)$
$= k(N \ln N - N_+ \ln N_+ - N_- \ln N_-)$
$= k\left(N \log N - \frac{N}{2}(1-\epsilon)\ln\left[\frac{N}{2}(1-\epsilon)\right] - \frac{N}{2}(1+\epsilon)\ln\left[\frac{N}{2}(1+\epsilon)\right]\right)$
$= k\left\{N \ln N - \frac{N}{2}(1+\epsilon)\ln N - \frac{N}{2}(1+\epsilon)\ln\frac{(1-\epsilon)}{2} - \frac{N}{2}(1+\epsilon)\ln N - \frac{N}{2}(1+\epsilon)\ln\frac{(1+\epsilon)}{2}\right\}$
$= k\left\{-\frac{N}{2}(1+\epsilon)\ln\frac{(1-\epsilon)}{2} - -\frac{N}{2}(1+\epsilon)\ln\frac{(1+\epsilon)}{2}\right\}$
$= \boxed{-\frac{kN}{2}\ln\left[\left(\frac{1-\epsilon}{1}\right)^{1-\epsilon}\left(\frac{1+\epsilon}{2}\right)^{1+\epsilon}\right] = S}$
Compute T
$\frac{1}{T} = \frac{\delta S}{\delta E}$
$= \left(\frac{\delta E}{\delta \epsilon}\right)^{-1}\frac{\delta S}{\delta E}$
$= \frac{1}{\mu P_m N} \times \frac{\delta}{\delta \epsilon}\left[-\frac{Nk}{2}\left((1-\epsilon)\ln(1-\epsilon) + (1+\epsilon)\ln(1+\epsilon) - 2\ln 2\right)\right]$
$= -\frac{k}{2\mu P_m}(-P(1-\epsilon) - 1 + \mathrm{P_m}(1_\epsilon) + 1)$
$\Rightarrow \boxed{\frac{1}{T} = \frac{k}{2\mu P_m}\ln\left(\frac{1-\epsilon}{1+\epsilon}\right)}$

Plot $T(\epsilon), S(\epsilon)$



1. $\epsilon > 0 \Rightarrow$ problem, T<0 imphyiscal (though maths correct)
   We did not take into account the possibility that particles MOVE
   H should contain kinetic and potential terms

ϵ MUST be negative

2. With this Caviat fixed

   S(T) becomes monotonic

3. Third law of thermodynamics is reproduced

When

$\epsilon \to -1$

$N_+ \to N$

There is only 1 state

$\{+,+,\dots,+,+\}$

$\ln 1 = 0$

This is useful because quantized(=discrete) states

Classical system

Phase-space:

$$\Gamma = \left\{\{P_{mk}, q_{mj}\}, m = 1, \dots, N, k, j = 1,2,3\right\}$$

P_{mk} 3N matrix

q_{mj} 3N coordinates

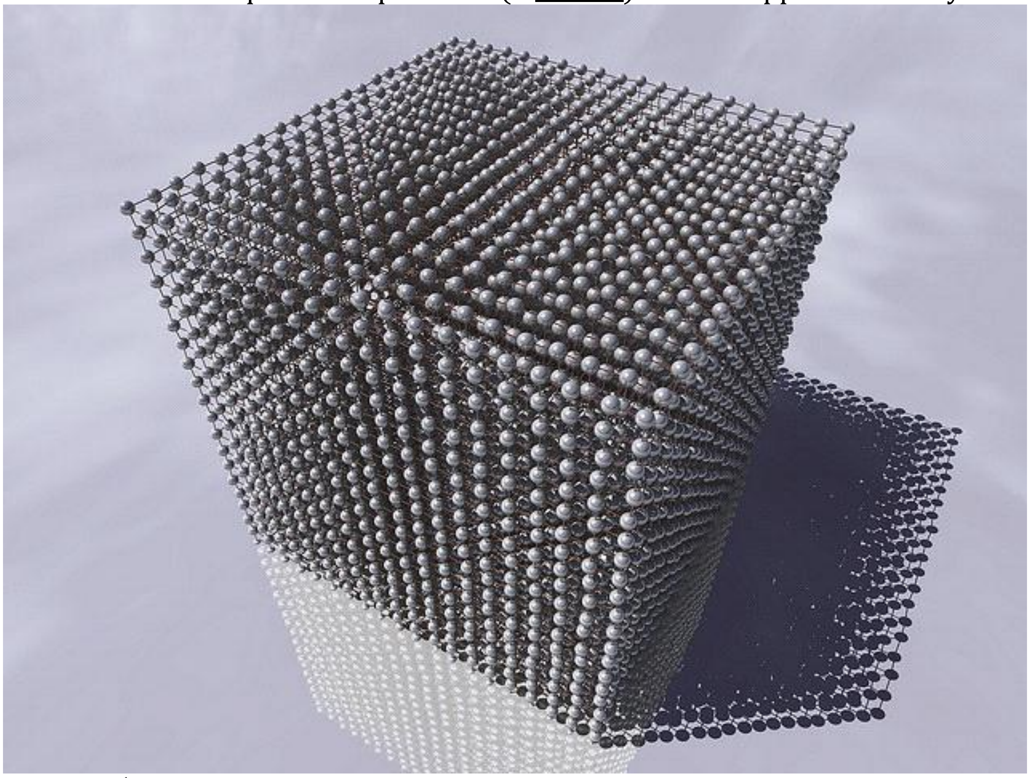
$$H = \sum_{i=1}^{N} H_i$$

$$H_i = \frac{1}{2m}\sum_{k=1}^{3} P_{ik}^2 + \frac{1}{2}m\omega^2\sum_{k=1}^{3} q_{ik}^2$$

3N harmonic oscillators

Approximate model of a solid:

Osc around equilibrium positions (if <u>SMALL</u>) are well approximated by harmonic oscillator



$$\Sigma(E) = \frac{1}{h^{3N}}\int_{H<E} d^{3N}q\,d^{3N}p$$

1. No 1/N! because particles can be distinguished by their "lattice" position
2. You could compute $\Omega(E)$

$$\Sigma(E) = \frac{1}{h^{3N}}\int_{H<E} d^{3N}q\,d^{3N}p$$

$$\frac{1}{2m}\sum_{i=1}^{3} P_i^2 + \frac{1}{2}m\omega^2\sum_{i=1}^{3} q_i^2$$

Do a change of variable in the integration!
Define 6N variables x_i

$$x_i = \begin{cases} \frac{1}{\sqrt{2m}} P_i \; when \; i = 1, \ldots, 3N \\ \sqrt{\frac{m\omega^2}{2}} q_{i-3N} \; when \; i = 3N+1, \ldots, 3N \end{cases}$$

A point in Γ is $\{p_1, p_2, p_3, \ldots, q_1, q_2, q_3, \ldots\}$
$p_1, p_2, p_3, \ldots = x, \ldots, x_{3N}$
$q_1, q_2, q_3, \ldots = x_{3N+1}, \ldots, x_{6N}$

The contraint reads

$$\boxed{\sum_{i=1}^{6N} x_i^2 \leq E}$$

6N dimensional ball with radius $R = \sqrt{E}$

$$\Sigma(E) = \frac{1}{h^{3N}} \int_{H<E} d^{3N}q d^{3N}p$$

$$= \frac{1}{h^{3N}} \int_{\sum x^2 \leq E} d^{6N}x \left(\sqrt{2m}\right)^{3N} \left(\sqrt{\frac{2}{m\omega^2}}\right)^{3N}$$

$$\left(\sqrt{2m}\right)^{3N} \left(\sqrt{\frac{2}{m\omega^2}}\right)^{3N} = jacobian$$

$$dp_i = dx_i \sqrt{2m}$$

$$= \left(\frac{2}{h\omega}\right)^{3N} vol\left(B^{6N}\left(R = \sqrt{E}\right)\right)$$

$$= \left(\frac{2}{h\omega}\right)^{3N} * \frac{\pi^{\frac{6N}{2}} R^{6N}}{\Gamma_E\left(1 + \frac{6N}{2}\right)}$$

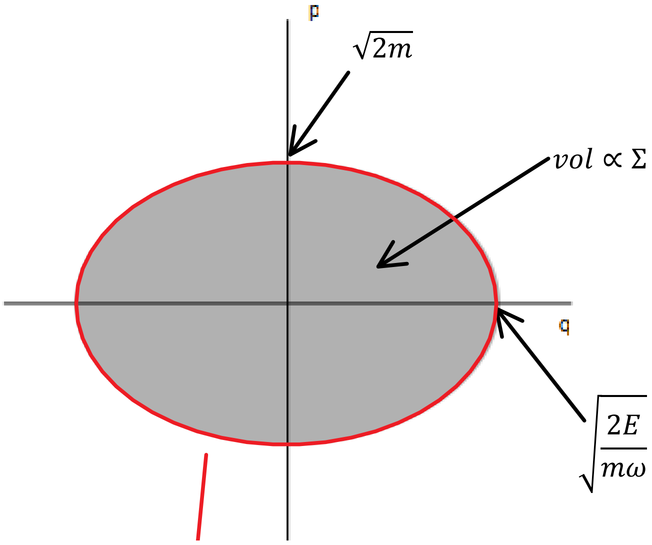
$$= \left(\frac{2\pi E}{h\omega}\right)^{3N} * \frac{1}{(3N)!}$$

$6N \rightarrow 2$
(i.e. suppose you have only one harmonic oscillator)

$$\Gamma = \left\{ \begin{Bmatrix} q \\ p \end{Bmatrix} \right\}$$

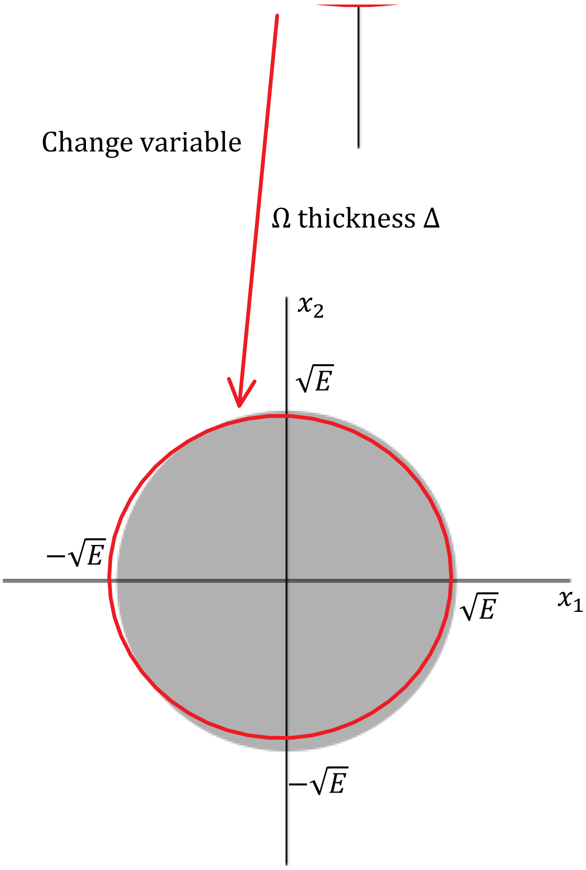
$$H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2 \leq E$$

Ellipse



$\sqrt{m\omega}$

Change variable



$x_{6N} = \sqrt{\frac{m\omega^2}{2}} q_{3N}$

$dx_{6N} = \sqrt{\frac{m\omega^2}{2}} dq_{3N}$

$dq_{3N} = \sqrt{\frac{2}{m\omega^2}} dx_{6N}$

$dp_{3N} = \sqrt{2m} dx_{6N}$?

$\Sigma = \left(\frac{2\pi E}{h\omega}\right)^{3N} * \frac{1}{(3N)!}$

Compote S (for large N)

Write all equations/definitions beforehand- helpful

$\underline{S \equiv k \ln \Sigma(E)}$

$= k \ln\left[\left(\frac{2\pi E}{h\omega}\right)^{3N} * \frac{1}{(3N)!}\right]$

Stirling

$\ln 3N! \sim 3N \ln 3N - 3N$

$= 3Nk \ln\left(\frac{2\pi E}{h\omega}\right) - k(3N \ln 3N - 3N)$

$= 3Nk\left[1 + \ln\left(\frac{2\pi E}{h\omega}\right)\right] = S$

$E = \frac{3Nh\omega}{2\pi e} \exp\left[\frac{S}{3Nk}\right]$

$T = \frac{\delta E}{\delta S} = \frac{E}{3Nk} \Rightarrow \boxed{E = 2NkT}$

$C_v = \frac{\delta E}{\delta T} = 3Nk$

Equipartitions in H there are

$3N\ q_i, and\ 3N\ p_i$

Entering quadratically

$$C_V = (3N + 3N)\frac{1}{2}k = 3Nk$$

In S, replace E=3NkT

$$S = 3Nk(1 + \ln\left(\frac{kT}{\hbar\omega}\right)$$

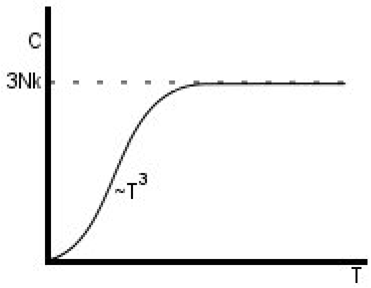
$$\hbar = \frac{h}{2\pi}$$

4. Violations of equipartition theorem

Example: Solid

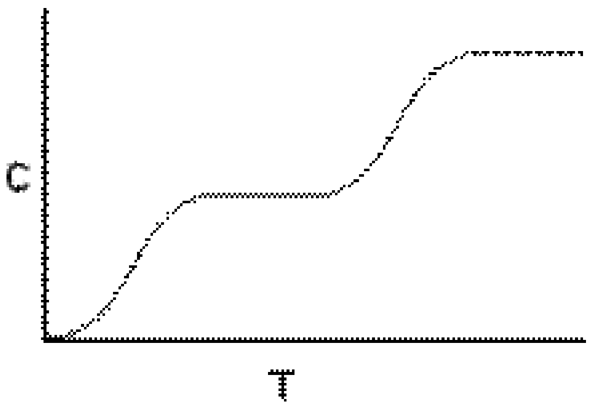
$C_v = 3NK$

Dulong-petit law



Solution involves QM

Example 2: Diatomic gas



No classical explanation (QM needed)

One can at best produce classical models that work well over ranges of T

1) $H = \frac{1}{2M}\Sigma_i P_i^2$

$P: c.m. momentum$

2) Rotation D. O. F.

$$H = \frac{1}{2M}\Sigma_i P_i^2 + \frac{1}{2I}\Sigma_i e_i^2$$

I: momentum of inertia

e: angular momentum

Equipartition

$$C_V = N \times \frac{k}{2} \times (3+2)$$
$$= \frac{5}{2}Nk$$

3) $H = \frac{1}{2M}\Sigma_i P_i^2 + \frac{1}{2I}\Sigma_i e_i^2 + \frac{1}{2\mu}\Sigma_i p_i + \frac{1}{2}\mu\omega^2\Sigma_i (R_i - R_0)^2$

μ: reduced mass

R_i:distance between atoms

(R_0 average)

p_i:momentum conjugate to R_i

Equipartition

$$C_V = \frac{1}{2}Nk(3+2+1+1)$$
$$= \frac{7}{2}Nk$$

Concept of energy threshold (QM) in order to connect 1-2-3

Microcanonical ensemble (classical)

Good

1) 1st, 2nd law
2) EQ state computed
3) At ordinary T, Equipartition theorem works

Bad

1) What is h?
2) $\frac{1}{N!}$ boltzmann factor?
3) Third law?
4) Exp. Violations of equipartition

Ugly

1) Calculations are hard!
2) N, E fixed Unphysical!

CANONICAL ENSEMBLE

Addresses 2) ugly feature and makes calculations more accessible!

Partition Function

Classical canonical ensemble

Assumptions

1. Physical systems of interest are in contact (equilibrium) with a heat reservoir at temperature T
2. N fixed

NO REFERENCE TO E

Density function

$$\rho = (q,p) \equiv e^{-\frac{1}{kT}H(q,p)}$$
$$= e^{-\beta H(p,q)}$$
$$\beta = \frac{1}{kT}$$

Partition function

$$Z_N = \frac{1}{h^{3N}N!}\int_\Gamma d^{3N}q d^{3N}p e^{-\beta H}$$

$\Gamma \leftarrow$ integral over whole phase-space

$\frac{1}{h^{3N}N!} = \textit{same as microcanonical}$

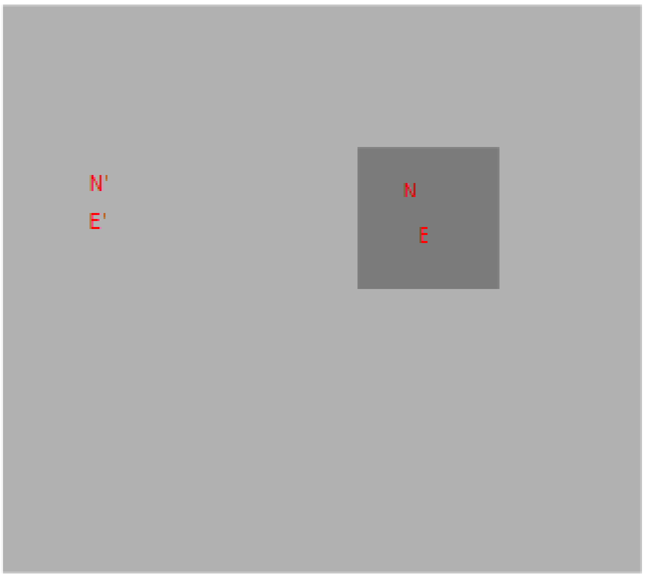
Define Helmoltz free energy as

$Z_N \equiv e^{-\beta F(v,T)}$

Show that

1. F is extensive
2. F satisfies $\boxed{F = U - TS}$

Derivation:
System N,E
Reservoir N', E'
$E' \gg E$
$N' \gg N$



$E_T = E + E'$
$N_T = N + N'$
USE Microcanonical

$$\Omega(E_T) = \int_{E_T<H<E_T+2\Delta} d^{3N}q d^{3N}p d^{3N}q' d^{3N}p'$$

Assume particles distinguishable and h=1
$d^{3N}q d^{3N}p = coordinates\ and\ momentua\ of\ system$
$d^{3N}q'^{d^{3N}}p' ='' \ of\ resevoir$

$$= \sum_E \int_{E<H<E+\Delta} d^{3N}q d^{3N}p \int_{E'<H<E'+\Delta} d^{3N}q' d^{3N}p'$$

$$= \sum_E \int_{E<H<E+\Delta} d^{3N}q d^{3N}p\, \Omega'(E')$$

$$= \sum_E \int_{E<H<E+\Delta} d^{3N}q d^{3N}p\, \Omega'(E_T - H)$$

$We\ know\ that$
$S'(E_T - H) = k \ln \Omega'(E_T - H)$

Taylor expand

$$= k \ln \Omega'(E_T) - k \frac{\delta \ln \Omega'(E_T)}{\delta E_T} H + \cdots$$

$$\cong S'(E_T) - \frac{\delta S'}{\delta E_T} H$$

$$= S'(E_T) - \frac{1}{T} H$$

$$\Omega'(E_T - H) = \exp\left[\frac{1}{k} S'\right]$$

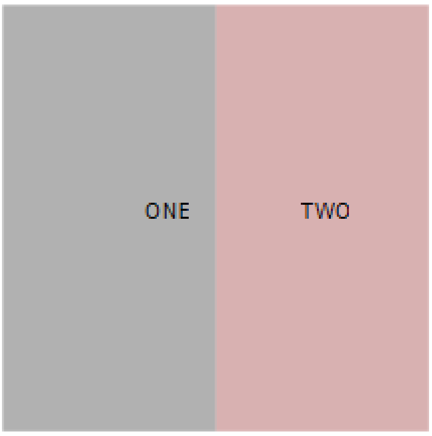
$$= \exp\left[\frac{S(E_T)}{k}\right]\exp\left[-\frac{H}{kT}\right]$$

$$\Omega(E_T) = \exp\left[\frac{S(E_T)}{k}\right]\sum_E \int_{E<H<E+\Delta} d^{3N}q d^{3N}p\, e^{-\frac{H}{kT}}$$

$$= \exp\left[\frac{S(E_T)}{k}\right]\int_\Gamma d^{3N}q d^{3N}p\, e^{-\frac{H}{kT}}$$

$$\exp\left[\frac{S(E_T)}{k}\right] = constand\ that\ \ is\ not\ interesting\ for\ our\ system$$

Essentive:
Proof:



Take h=1

$$N = N_1 + N_2$$

Same temperature T

$$Z_N = \frac{1}{N_1!}\frac{1}{N_2!}\int d^{3N}q d^{3N}p\, e^{-\beta H}$$

$$H = H_1 + H_2$$

As long as the interactions are short ranged, H=H_1+H_2

$$= \frac{1}{N_1!\,N_2!}\int d^{3N}q_1 d^{3N}p_1 d^{3N}q_2 d^{3N}p_2\, e^{-\beta H_1} e^{-\beta H_2}$$

$$= \left[\frac{1}{N_1!}\int d^{3N}q_1 d^{3N}p_1\, e^{-\beta H_1}\right] \times \left[\frac{1}{N_2!}\int d^{3N}q_2 d^{3N}p_2\, e^{-\beta H_2}\right]$$

$$= Z_{N_1} Z_{N_2}$$

$$F = -kT \ln Z_N$$

$$= -kT \ln Z_{N_1} Z_{N_2}$$

$$= -kT \ln Z_{N_1} - kT \ln Z_{N_2}$$

$$= F_1 + F_2$$

Prove that $F = U - TS$

We defined

$$e^{-\beta f} \equiv \frac{1}{h^{3N}N!}\int d^{3N}q d^{3N}p\, e^{-\beta H}$$

Rewrite as

$$1 = \frac{1}{h^{3N}N!}\int d^{3N}q d^{3N}p\, e^{-\beta(H-F)}$$

Take derivative in respect to β

$$0 = \frac{1}{h^{3N}N!}\int d^{3N}q d^{3N}p\, e^{-\beta(H-F)}\ \frac{\delta}{\delta\beta}\big(-\beta(H-F)\big)$$
$$= \frac{1}{h^{3N}N!}\int d^{3N}q d^{3N}p\, e^{-\beta(H-F)} \times \left[F - H + \beta\frac{\delta F}{\delta\beta}\right]$$
Internal energy can (must) be defined be ensemble average
$$E = U =< H >$$
$$= \int \frac{d^{3N}q d^{3N}p}{h^{3N}N!} e^{-\beta(H-F)} H$$
Remember
$$< p > \ = \frac{\int d \ldots p \times f}{\int d \ldots \rho}$$
Rewrite
$$0 = \left(F + \beta\frac{\delta F}{\delta\beta}\right) \times \frac{1}{h^{3N}N!}\int d^{3N}q d^{3N}p\, e^{-\beta(H-F)} - \frac{1}{h^{3N}N!}\int d^{3N}q d^{3N}p\, e^{-\beta(H-F)}$$
$$= \left(F + \beta\frac{\delta F}{\delta\beta}\right) \times \frac{1}{h^{3N}N!}\int d^{3N}q d^{3N}p\, e^{-\beta(H-F)} - U$$
$$0 = F - U + \beta\frac{\delta F}{\delta\beta}$$
$$\beta = \frac{1}{kT}$$
$$\beta\frac{\delta F}{\delta\beta} = \beta\left(\frac{\delta\beta}{\delta T}\right)^{-1}\frac{\delta}{\delta T}F$$
$$= \frac{1}{kT}\left(-\frac{1}{kT^2}\right)^{-1}\frac{\delta F}{\delta T}$$
$$= -T\frac{\delta F}{\delta T}$$
$$0 = F - U - T\frac{\delta F}{\delta T}$$
Maxwell
$$S = -\frac{\delta F}{\delta T}$$
$$\Rightarrow \boxed{F = U - TS}$$

Canonical ensemble

23 November 2011
09:10

Prescription

1. Compute
$$Z_N = \frac{1}{h^3 N!}\int_\Gamma d^{3N}q\, d^{3N}p\, e^{-\beta H}$$
$N! =$ indistinguishable particles
$\beta = \frac{1}{kT}$
2. Compute Free energy
$F = -kT \ln Z_N$
3. Maxwell relations
$S = -\frac{\delta F}{\delta T}$
$P = -\frac{\delta F}{\delta V}$
4. Thermodynamics

Exercise N.1: monoatomic ideal gas

$$H = \frac{1}{2m}\sum_{i=1}^{3N} P_i^2$$
N particles
V volume
$$\int_{-\infty}^{\infty} dx\, e^{-\gamma x^2} = \sqrt{\frac{\pi}{\gamma}}$$

Set h=1 for simplicity
$$Z_N = \frac{1}{N!}\int_\Gamma d^{3N}q\, d^{3N}p\, e^{-\beta H}$$

$$= \frac{1}{N!}\int_\Gamma dq_1 dq_2 \dots dq_{3N} dp_1 dp_2 \dots dp_{3N}\, e^{-\frac{\beta}{2m}\Sigma_i P_i^2}$$
$$= \frac{1}{N!} * V^N * \int_{-\infty}^{\infty} dp_1 e^{-\frac{\beta}{2m}p_1^2} * \int_{-\infty}^{\infty} dp_2 e^{-\frac{\beta}{2m}p_2^2} * \int_{-\infty}^{\infty} dp_3 e^{-\frac{\beta}{2m}p_3^2}$$
$$= \frac{V^N}{N!}\left[\int_{-\infty}^{\infty} dp\, e^{-\frac{1}{2mkT}p^2}\right]^{3N} = \frac{V^N}{N!}\left(\sqrt{2\pi mkT}\right)^{3N}$$
$$= \frac{V^N T^{\frac{3N}{2}} C^{\frac{3N}{2}}}{N!}$$
C=arbitrary constant

$$F = -kT \ln Z_N$$
$$= -kT \ln \frac{V^N (cT)^{\frac{3}{2}N}}{N!}$$
$$= -NkT \ln\left[\frac{V}{N} cT^{\frac{3}{2}}\right]$$
$$P = -\frac{\delta F}{\delta V} = -(-NkT)\frac{1}{V} = \frac{NkT}{V} \Rightarrow \boxed{PV = NkT}$$

$$S = -\frac{\delta F}{\delta T} = Nk \ln\left(\frac{V}{N} cT^{\frac{3}{2}}\right) + NkT\left(\frac{3}{2}\frac{1}{T}\right)$$

$= \frac{F}{T} + \frac{3}{2}Nk$
$E = U = F + TS$
$= F + T\left(-\frac{F}{T} + \frac{3}{2}Nk\right)$
$= F - F + \frac{3}{2}NkT \Rightarrow \boxed{U = \frac{3}{2}NkT} \Rightarrow \boxed{C_v = \frac{\delta U}{\delta T} = \frac{3}{2}Nk}$

1. As long as N large, Microcanonical and canonical ensembles yield same thermodynamics
2. Be precise at early stages

Exercise N.2 Diatomic ideal gas
(Room T)
$H = \sum_i \frac{1}{2m}P_i^2 + \frac{1}{2I}l_i^2$
Where phase-space is
$\Gamma = \left\{\{\overline{Q_i}, \overline{P_i}, \theta_i, \phi_i \overline{l_i}\}, i = 1 \ldots N\right\}$
$\theta_i, \phi_i \rightarrow (h = 1)\ decribing\ rotation, not\ d\theta d\phi$
$\overline{l_i}\ only\ two\ ways\ of\ rotation$
???
$Z_N = \frac{1}{N!}\int d^{3N}Q\ d^{3N}P d^N \cos\theta\ d^N\phi\ d^{2N}l\ e^{-\beta H}$
$= \frac{1}{N!}V^N(4\pi)^N$
$4\pi = angles$
Why 2N
N particles $\bar{l}$ vector $\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$
But diatomic∴ 2 meaningful components, so 2N
$Z_N = V^N(4\pi)^N \int dP_1 dP_2 \ldots dP_{3N} dl_1 dl_2 \ldots dl_{2N} \exp -\beta\left[\sum_i \frac{1}{2m}P_i^2 + \sum_i \frac{1}{2I}l_i^2\right]$
$= \frac{1}{N!}V^N(4\pi)^N\left[\int dP\ e^{-\frac{1}{2mkT}P^2}\right]^{3N}\left[\int dle^{-\frac{1}{IkT}l^2}\right]^{2N}$
$= \frac{1}{N!}v^N(4\pi)^N(2\pi mkT)^{\frac{3N}{2}}(2\pi IkT)^{\frac{3N}{2}}$
$Z_N = \frac{V^N(cT)^{\frac{5N}{2}}}{N!}$
$F = -kT\ln Z_N$
$= -kT\ln\left(\frac{V^N(cT)^{\frac{5N}{2}}}{N!}\right)$
$\ln N! \approx N\ln N - N$
$= -NkT\ln\left(\frac{V^N(cT)^{\frac{5}{2}}}{N}\right) - N \cong -NkT\ln\left(\frac{V^N(cT)^{\frac{5}{2}}}{N}\right)$

$P = -\frac{\delta F}{\delta V} = \frac{NkT}{V} \Rightarrow \boxed{PV = NkT}$
$S = -\frac{\delta F}{\delta T} = -\frac{F}{T} + NkT\left(\frac{5}{2}\frac{1}{T}\right)$
$\boxed{C_v = \frac{5}{2}Nk}$

QM Canonical ensemble

24 November 2011
09:07

Pragmatic approach
Classical
Phase-space Γ
QM

- Discrete <u>energy levels</u>
    $\hat{\epsilon}_k$, occupation number $\hat{m}_k$
    Sequence of $\{\hat{m}_k\}\ replaces\ \Gamma$
- $E = \Sigma_k \hat{m}_k \hat{\epsilon}_k$
    Sum of levels
- $\Omega(\epsilon) = \int_{\Gamma(E<H<E+\Delta)} d^{3N}q\, d^{3N}p \leftrightarrow \Omega(\epsilon) = \sum_{\{\hat{m}_k\}} W(\{\hat{m}_k\})$
    W: number of different ways to realize/write same state

Example: 2 state system

N_+, N_-

$W = \binom{N}{N_+}$

Harmonic oscillator (QM)

$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$

Eigenstates

$\hat{\epsilon}_k = \left(k + \frac{1}{2}\right)\hbar\omega$

$k \in N_0$

(no degeneracies)

Exercise: compute canonical ensemble for system of 3N oscillators which are decoupled and have all same ω

(Einstein solid)

Compute partition function

$$Z_N = \sum_{m_1=0}^{\infty} \exp(-\beta\hat{\epsilon}_{m_1}) * \sum_{m_2=0}^{\infty} \exp(-\beta\hat{\epsilon}_{m_2}) * \cdots * \sum_{m_{3N}=0}^{\infty} \exp(-\beta\hat{\epsilon}_{3N})$$

Because ω is same, AND decoupled oscillators

$$= \left[\sum_{(m=0)}^{\infty} \exp\left(-\beta\left(m + \frac{1}{2}\right)\hbar\omega\right)\right]^{3N}$$

Because no degeneracies, summation can be done explicitly

$$= \left[\exp\left(-\frac{\beta\hbar\omega}{2}\right)\right]^{3N} \left[\sum_{m=0}^{\infty} \exp(-\beta m\hbar\omega)\right]^{3N}$$

$$= \left[\exp\left(-\frac{\beta\hbar\omega}{2}\right)\right]^{3N} [1 + \exp(-\beta\hbar\omega) + (\exp(-\beta\hbar\omega))^2 + \cdots]^{3N}$$

$\beta\hbar\omega > 0 \Rightarrow \exp(-\beta\hbar\omega) < 1$

Geometric series

$$= \left[\exp\left(-\frac{\beta\hbar\omega}{2}\right)\right]^{3N} \left[\frac{1}{1 - \exp[-\beta\hbar\omega]}\right]^{3N}$$

$$= \left[\frac{\exp\left(-\frac{\beta\hbar\omega}{2}\right)}{\exp\left(-\frac{\beta\hbar\omega}{2}\right)\left(\exp\left(\frac{\beta\hbar\omega}{2}\right) - \exp\left(-\frac{\beta\hbar\omega}{2}\right)\right)}\right]^{3N}$$

$$= \left[\frac{1}{\left(\exp\left(\frac{\beta\hbar\omega}{2}\right) - \exp\left(-\frac{\beta\hbar\omega}{2}\right)\right)}\right]^{3N}$$

$$= \left[\frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}\right]^{3N} = Z_N$$

Free energy

$$F = -kT \ln Z_N$$
$$= 3NkT \ln\left[2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right]$$
$$S = -\frac{\delta F}{\delta T}$$
$$= -\frac{\delta}{\delta T}\left(3NkT \ln\left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right]\right)$$
$$= -3Nk \ln\left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right] - 3NkT\frac{1}{2\sinh\frac{\hbar\omega}{2kT}} 2\cosh\frac{\hbar\omega}{2kT} * \left(-\frac{\hbar\omega}{kT^2}\right)$$
$$= -3Nk \ln\left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right] + \frac{3}{2}\frac{N\hbar\omega}{T}\coth\left(\frac{\hbar\omega}{2kT}\right)$$
$$S_{classical} = 3Nk\left[1 + \ln\frac{kT}{\hbar\omega}\right]$$

From classical calculation

Take several limits and compare

$$T \to \infty \quad x = \frac{\hbar\omega}{2kT} \to 0$$
$$\coth x \sim \frac{1}{x} \quad for\ x \to 0$$
$$\sinh x \sim x \quad for\ x \to 0$$
$$S_{QM} \approx_{large\ T} -3Nk \ln\frac{\hbar\omega}{kT} + \frac{3}{2}\frac{N\hbar\omega}{T}\frac{2kT}{\hbar\omega}$$
$$= -3Nk \ln\frac{\hbar\omega}{kT} + \frac{3Nk}{T}$$
$$= -3Nk\left[\ln\frac{kT}{\hbar\omega} + 1\right]$$

What about $T \to 0$

Then $x \to \infty$

And $\begin{cases} \coth x \to 1 \\ \sinh x \to \frac{e^x}{2} \end{cases}$

$$S_{QM} \approx_{small\ T} -3Nk \ln\left[2 * \frac{\exp\left[\frac{\hbar\omega}{2kT}\right]}{2}\right] + \frac{3}{2}\frac{N\hbar\omega}{T} * 1$$
$$= -3Nk\ \frac{\hbar\omega}{2kT} + \frac{3}{2}\frac{N\hbar\omega}{T}$$
$$= -\frac{3}{2}\frac{N\hbar\omega}{T} + \frac{3}{2}\frac{N\hbar\omega}{T} = 0$$

THIRD LAW!

30 November 2011
09:10

Internal energy

$$S = -\frac{F}{T} + \frac{3N\hbar\omega}{2t}\coth\frac{\hbar\omega}{2kT}$$

Compute

$$U = E = F + TS$$

$$= F + T\left(-\frac{F}{T} + \frac{3N\hbar\omega}{2t}\coth\frac{\hbar\omega}{2kT}\right)$$

$$= \frac{3}{2}N\hbar\omega\coth\left(\frac{\hbar\omega}{2kT}\right)$$

For $T \to \infty$

$$\frac{\hbar\omega}{2kT} \to small$$

$$\coth x \sim \frac{1}{x} \quad x \ll 1$$

$$\Rightarrow \coth\frac{\hbar\omega}{2kT} \to \frac{2kT}{\hbar\omega}$$

$$U = \frac{3}{2}N\hbar\omega\coth\frac{\hbar\omega}{2kT}$$

$$T \approx_{large} \frac{3}{2}N\hbar\omega\frac{2kT}{\hbar\omega} = 3NkT$$

This agrees with classical calculation and equipartition theorem

When $T \to 0$

$$\coth x \to 1 \; when \; x \to \infty$$

Hence

$$U \approx_{t-small} \frac{3}{2}N\hbar\omega \times (1)$$

$$= \frac{3}{2}N\hbar\omega$$

Remember

$$\hat{\epsilon}_k = \hbar\omega\left(k + \frac{1}{2}\right)$$

$$k = 0,1,2,\ldots$$

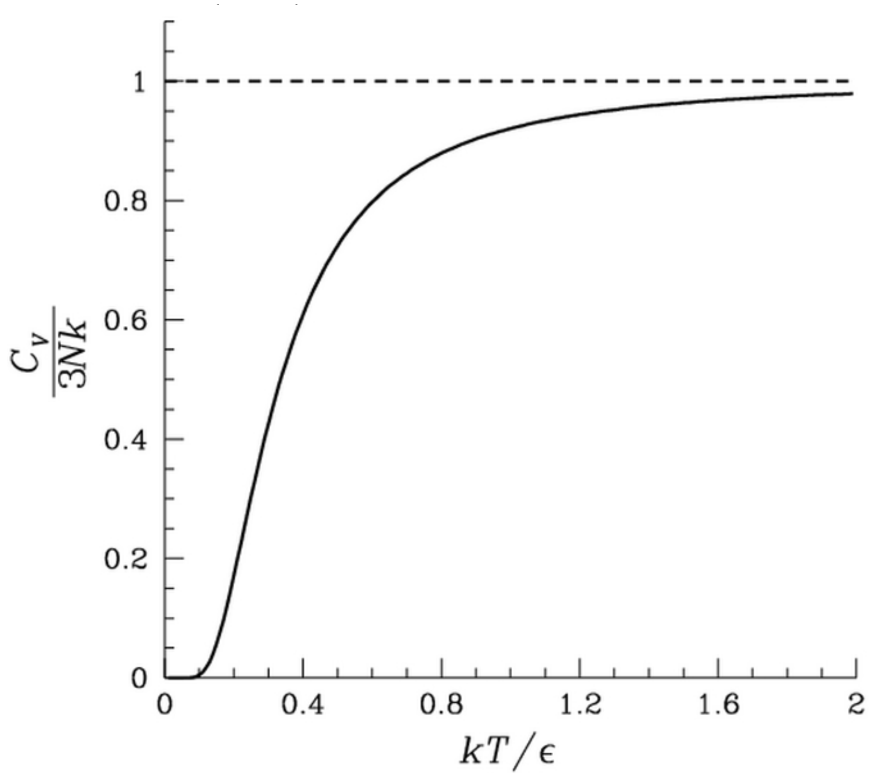
$$\hat{\epsilon}_0 = \frac{\hbar\omega}{2}$$

At very low T

At very low T, virtually all the oscillators are in the ground state!

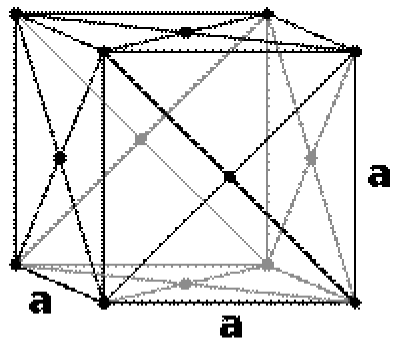
Also

$$C_v = \frac{\delta U}{\delta T} = \begin{cases} 3Nk \; (T \to \infty) \\ 0 \; (T \to 0) \end{cases}$$



$$C_v = 3Nk\left(\frac{\frac{\hbar\omega}{2kT}}{\sinh\frac{\hbar\omega}{2kT}}\right)^2$$

Good: QM calculation explains the fact that equipartition does not hold at low T



In a real solid, the interactions (i.e. potential) cannot be reduced to 3N copies of oscillator , All with same frequency ω! Many different values of ω to be used. (interaction between all particles)

In practice: our H is too simple to fit the data

Grand Canonical ensemble

30 November 2011
09:31

Remember that:

Density function ρ chosen on the basis of a set of constraints which define ensemble

$$\begin{array}{ccc} Microcanonical & \rightarrow & Canonical \\ E, N\ fixed & & T, N\ Fixed \\ \Downarrow & & \Downarrow \\ \rho = \begin{cases} 1 & E < H < E + \Delta \\ 0 & otherwise \end{cases} & & \rho = e^{-\beta H} \end{array}$$

What does $e^{-\beta H}$ mean?

- At fixed T (β) states with lower energy are more important
- At lower T, the difference in the weight is very large

β: parameter encodes thermal ("disorder") fluctuations
H: classical dynamics
However: Keeping N fixed is unrealistic!

<u>Grand Canonical</u>

Constraints are (T, μ) are fixed

μ =chemical potential

Replaces the role of N in the construction

In general, we expect

$$F = F(T, V, N)$$

$$\mu \equiv \left(\frac{\delta F}{\delta N}\right)_{V,T}$$

Example

N_2 H_2

$$N_2 + 3H_3 \rightarrow 2NH_3$$

Generalize thermodynamic potentials

$$dF = -S\, dT - P\, dV + \mu\, dN$$
$$dU = T\, dS - P\, dV + \mu\, dN$$

<u>Grand Canonical Ensemble</u>

N allowed to vary

Define density function

$$\rho(N, q, p) \equiv \frac{z^N}{N!\, h^{3N}} \exp[-\beta PV - \beta H]$$

N=discrete parameter

p,q=for each choice of there are 3N q_i and 3N p_i

P=pressure

Fugacity

$$z \equiv e^{\beta\mu}$$

$$\beta \equiv \frac{1}{kT}$$

μ= chemical potential

Define Grand Partition Function

$$\mathcal{Z} \equiv \sum_{N=0}^{\infty} z^N Z_N$$

Z_N =partition function

Equation of state

$$\ln \mathcal{Z} = \frac{PV}{kT}$$

All other interesting thermodynamic quantities compute via ensemble averages

Derivation

2 systems

$N_1 \ll N_2 = N - N_1$

$V_1 \ll V_2 = V - V_1$

Systems at equilibrium with each other

Same T

But in general, we allow them to exchange particles

Partition function of (1)+(2) (two systems)

(set h=1)

$$Z_N = \int \left(\frac{d^{3N}q d^{3N}p}{N!}\right) e^{-\beta H}$$

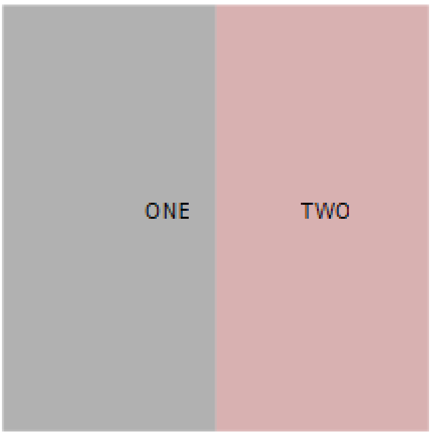
Assume that $H = H_1 + H_2$

(no interaction terms between 1 and 2)

$$= \sum_{N_1=0}^{N} \frac{N!}{N_1!\,N_2!} \int \left(\frac{d^{3N}q_1 d^{3N}p_1 d^{3N}q_2 d^{3N}p_2}{N!}\right) e^{-\beta(H_1+H_2)}$$

$d^{3N}q d^{3N}p$

$= dq_1 dq_2 dq_3 \dots dq_{3N_1} dq_{3N+1} \dots dq_{3N} \times dp_1 dp_2 dp_3 \dots dp_{3N_1} dp_{3N+1} \dots dp_{3N}$



$dq_1 dq_2 dq_3 \dots dq_{3N_1}$ and $dp_1 dp_2 dp_3 \dots dp_{3N_1}$ correspond to 1, others correspond to two

$$= \sum_{N_1=0}^{N} \int \frac{d^{3N_1}q_1 d^{3N_1}p_1}{N_1!} e^{-\beta H_1} \int \frac{d^{3N_2}q_2 d^{3N_2}p_2}{N_2!} e^{-\beta H_2}$$

$N_2 = N - N_1$

$$= \sum_{N_1=0}^{N} \int \frac{d^{3N_1}q_1 d^{3N_1}p_1}{N_1!} e^{-\beta H_1} Z_{N-N_1} = Z_N$$

$$1 = \sum_{N_1} \int \frac{d^{3N_1}q_1 d^{3N_1}p_1}{N_1!} e^{-\beta H_1} \frac{Z_{N_2}}{Z_N}$$

$F_N \equiv kT \ln Z_N$

$$1 = \sum_{N_1=0}^{N} \int \frac{d^{3N_1}q_1 d^{3N_1}p_1}{N_1!} e^{-\beta H_1} e^{-\beta(F_{N_2}-F_N)}$$

Taylor expand for small $N_1 = N - N_2$

$F_{N_2} - F_N = F(V - V_1, T, N - N_1) - F(V, T, N)$

$\approx \frac{\delta F}{\delta V}(\Delta V) + \frac{\delta F}{\delta N}(\Delta N)$

$= (-P)(-V_1) + \mu(-N_1)$

$F_{N_2} - F_N = PV_1 - \mu N_1$

$$1 = \sum_{N_1=0}^{N} \int \frac{d^{3N_1}q_1 d^{3N_1}p_1}{N_1!} e^{-\beta H_1} e^{-\beta(PV_1 - \mu N_1)}$$

P has no index because systems are in mechanical equilibrium (same pressure)

$e^{-\beta(PV_1-\mu N_1)}$ does not depend on (q_1, p_1)

$$= \sum_{N_1=0}^{N} e^{-\beta(PV_1-\mu N_1)} Z_{N_1}$$

Now take $N \to \infty$ and drop the "1"

$$1 = \sum_{N=0}^{\infty} e^{-\beta PV} e^{\beta\mu N} Z_N$$

$e^{-\beta PV}$ = no N dependence

$$= e^{-\beta PV} \sum_{N=0}^{\infty} z^N Z_N$$

$z^N Z_N = Z$

(grand partition function)

$$= e^{-\beta PV} Z$$

$$\Rightarrow \boxed{\frac{PV}{kT} = \ln Z}$$

Average number of particles:

Def. ensemble average

$$< N > \ = \frac{\sum_N z^N Z_N N}{\sum_N z^N Z_N}$$

Because Z_N does not depend on z,

$$< N > \ = \frac{\frac{\delta}{\delta z} \sum_N z^N Z_N}{\sum_N z^N Z_N}$$

$$= \frac{\frac{\delta}{\delta z} Z}{Z}$$

$$= \frac{\delta}{\delta z} \ln Z$$

$$\boxed{< N > = z \frac{\delta}{\delta z} \ln Z(z, V, T)}$$

Internal energy

$$U = E = < H >$$

$$= \frac{\delta}{\delta\beta} \ln Z$$

Proof

$$-\frac{\delta}{\delta\beta} \ln Z = -\frac{\delta}{\delta\beta} \ln \sum_N z^N Z_N$$

$$= -\frac{\delta}{\delta\beta} \ln \sum_N z^N \int$$

''''''

$$= -(\frac{\delta}{\delta\beta} \left(\sum_N z^N \int \frac{d^{3N}q d^{3N}p}{N!} e^{-\beta H} \right)) / (\sum_N Z^N \int \frac{d^{3N}q d^{3N}p}{N!}$$

''''''

Monoatomic ideal gas

$$H = \frac{1}{2m} \sum_i P_i^2$$

We already know that

$$Z_N = \frac{\left(cVT^{\frac{2}{3}}\right)^N}{N!}$$

Also, let us compute

$$Z = \sum_{N=0}^{\infty} z^N Z_N$$

$$= \sum_{N=0}^{\infty} \frac{\left(czVT^{\frac{3}{2}}\right)^N}{N!}$$

$$= \exp\left[czVT^{\frac{3}{2}}\right]$$

Equation of state

$$\frac{PV}{kT} = \ln Z = czVT^{\frac{3}{2}}$$

Compute <N>

$$< N > \ = z\frac{\delta}{\delta z}\ln Z$$

$$= z\frac{\delta}{\delta z}\left(czVT^{\frac{3}{2}}\right)$$

$$= czvT^{\frac{3}{2}} = \ln Z$$

Go back to E.O. state

$$\frac{PV}{kT} = \ln Z = < N >$$

$$\Rightarrow \boxed{PV = < N > kT}$$

Internal energy

$$U = < H > \ = -\frac{\delta}{\delta \beta}\ln Z$$

$$= -\left(\frac{\delta \beta}{\delta T}\right)^{-1}\frac{\delta}{\delta T}\ln Z$$

$$= \left(-\frac{1}{kT}\right)^{-1}\frac{\delta}{\delta T}\left(czVT^{\frac{3}{2}}\right)$$

$$= \frac{3}{2}kT\ TczVT^{\frac{1}{2}}$$

$$= \frac{3}{2}kT\left(czVT^{\frac{3}{2}}\right)$$

$$= \boxed{\frac{3}{2} < N > kT = U}$$

Quantum Gas Distributions

Grand canonical treatment

Basic QM principles

1) Energy levels of many systems are <u>discrete</u>

e.g.

Particle in a box 1-D

$$\hat{s}_m = \frac{\hbar^2 m^2}{8mL}$$

Harmonic oscillator

$$\hat{E}_m = \hbar\omega\left(m + \frac{1}{2}\right)$$

…

2) Because identical particles are <u>not distinguishable</u> then

If 1- particle states are

$\psi_1(x_1)$

$\psi_2(x_1)$

The 2-particle state is $\boxed{NOT}$ $\psi_1(x_1)\psi_2(x_1)$

<u>How to build 2-particle states?</u>

08 December 2011
09:23

In nature, there exist two kinds of particles: Bosons, Fermions

- Bosons:
  - The state of N-particles is symmetric under exchange of any two particles
  - Example: $\psi(x_1, x_2) = \psi_1(x_1)$
  - ````

- Fermions:

  Example $\psi(x_1, x_2) = \psi_1(x_2)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)$
  Fermions obey pauli exclusion principle: two identical fermions cannot occupy the same state
  Proof
    "same state" means that the function
    $\psi_1(x) = \psi_2(x) \equiv \phi(x)$
    $\psi(x_1, x_2) = \phi(x_1)\phi(x_2) - \phi(x_2)\phi(x_1) = 0$

Spin statistics theorem fermions have half-integer spin, while bosons have integer spin
Fermions: electrons, quark, neutrinos, muon, protons, neutron,…
Bosons: photon, W,Z,Gluon

- We call level the single particle energy eigenstate
  $\hat{\varepsilon}_i : i\ label\ state$
  Total energy of N particles

  $$E = \sum_{k=1}^{N} \hat{\varepsilon}_{ik}$$

  i= state
  k=label of particle
- We call occupation number $\widehat{m}_i$ the number of particles in energy level $\hat{\varepsilon}_i$

  $$E = \sum_{i=0}^{\infty} \widehat{m}_i \hat{\varepsilon}_i$$

- We can describe a microscopic state in terms of the sequence of occupation numbers
  $\{\widehat{m}_k\} \equiv \{\widehat{m}_0, \widehat{m}_1, \widehat{m}_2, \dots, \widehat{m}_N \dots\}$

Consider a gas of particles occupying volume V at equilibrium with temerature T, and compute the Grand Canonical Ensemble

$$Z \equiv \sum_{N=0}^{\infty} z^N Z_N$$

N= number of functions
$z = e^{-\beta\mu}$ = fugacity
Z_N =Partition function

$$= \sum_N z^N \sum_{\substack{\{\widehat{m}_k\} \\ N}} e^{-\beta \sum_k \widehat{m}_k \hat{\varepsilon}_k}$$

$\sum_N$ = sum over different number of particle
$\sum_k$ =sum over levels
$\sum_{\substack{\{\widehat{m}_k\} \\ N}}$ =sum over all possible sequences of occupation number subject to the constraint $\sum_{k=0}^{\infty} \widehat{m}_k = N$

Our first task is to solve constraint!

$$= \sum_{N} \sum_{\substack{\{\hat{m}_k\} \\ N}} z^{\sum_k \hat{m}_k} e^{-\beta \sum_k \hat{m}_k \hat{\varepsilon}_k}$$

$$= \sum_{N} \sum_{\substack{\{\hat{m}_k\} \\ N}} e^{-\beta \sum_k \hat{m}_k (\hat{\varepsilon}_k - \mu)}$$

$$= \sum_{\{\hat{m}_k\}} e^{-\beta \sum_k \hat{m}_k (\hat{\varepsilon}_k - \mu)}$$

$\{\hat{m}_k\} \Leftarrow$no constraints

$$= \sum_{\{\hat{m}_k\}} \prod_{k} \left[e^{-\beta(\hat{\varepsilon}_k - \mu)}\right]^{\hat{m}_k}$$

$$= \sum_{\hat{m}_0} \sum_{\hat{m}_1} \sum_{\hat{m}_2} \dots \left[e^{-\beta(\hat{\varepsilon}_k - \mu)}\right]^{\hat{m}_0} \left[e^{-\beta(\hat{\varepsilon}_k - \mu)}\right]^{\hat{m}_1} \dots$$

$$= \prod_{k=0}^{\infty} \left[\sum_{\hat{m}_k=0}^{\infty} \left[e^{-\beta(\hat{\varepsilon}_k - \mu)}\right]^{\hat{m}_k} \right]$$

We managed to carefully rewrite Z as summations done level-by-level

Quantum gas

14 December 2011
09:16

Grand canonical ensemble

$$Z = \sum_N z^N Z_N$$

$$= \prod_{k=0}^{\infty} \left[\sum_{\widehat{m}_k=0}^{\infty} [\exp -\beta(\hat{\epsilon}_k - \mu)]^{\widehat{m}_k} \right]$$

k=levels

$\widehat{m}_k$ =occupation number of level

NO CONSTRAINTS

Fermion

$\widehat{m}_k = 0,1$

$$\sum_{\widehat{m}_k} \left[+e^{-\beta(\hat{\epsilon}_k-\mu)} \right]^{\widehat{m}_k} = \left[e^{-\beta(\hat{\epsilon}_k-\mu)} \right]^0 + \left[e^{-\beta(\hat{\epsilon}_k-\mu)} \right]^1 = 1 + e^{-\beta(\hat{\epsilon}_k-\mu)}$$

$$Z = \prod_{k=0}^{\infty} \left[1 + e^{-\beta(\hat{\epsilon}_k-\mu)} \right]$$

Equation of state

$$\frac{PV}{kT} = \ln Z = \ln \prod_{k=0}^{\infty} \left[1 + e^{-\beta(\hat{\epsilon}_k-\mu)} \right]$$

$$= \sum_k \ln\left(1 + e^{-\beta(\hat{\epsilon}_k-\mu)}\right)$$

k=levels

Occupation number $z^N = e^{\beta\mu}: Fugacity$

$$< N > \;\; = \sum_k < n_k >$$

Average total number of particles in terms of average occupation numbers

$$< N > \;\; = z \frac{\delta}{\delta z} \ln Z$$

$$= z \frac{\delta}{\delta z} \sum_k \ln\left(1 + z e^{-\beta\hat{\epsilon}_k}\right)$$

$$= \sum_k \left(z \frac{1}{1 + z e^{-\beta\hat{\epsilon}_k}} \right)$$

By comparing term by term

$$\boxed{< \widehat{m}_k > \;\; = \frac{1}{1 + e^{\beta(\hat{\epsilon}_k-\mu)}}}$$

Fermi-Dirac Distribution

Bosons

$\widehat{m}_k = 0,1,2,3, \ldots$

$$\sum_{\widehat{m}_k=0}^{\infty} \left(e^{-\beta(\hat{\epsilon}_k-\mu)}\right)^{\widehat{m}_k} = \left(e^{-\beta(\hat{\epsilon}_k-\mu)}\right)^0 + \left(e^{-\beta(\hat{\epsilon}_k-\mu)}\right)^1 + \left(e^{-\beta(\hat{\epsilon}_k-\mu)}\right)^2 + \left(e^{-\beta(\hat{\epsilon}_k-\mu)}\right)^3 + \cdots$$

GEOMETRIC SERIES!

$$= \frac{1}{1 - e^{-\beta(\hat{\epsilon}_k-\mu)}}$$

$$Z = \sum_k \frac{1}{1 - e^{-\beta(\hat{\epsilon}_k-\mu)}}$$

Eq state

$$\frac{PV}{kT} = \ln Z = -\sum_k \ln\left(1 - e^{-\beta(\hat{\epsilon}_k - \mu)}\right)$$

$$< N > \ = z\frac{\delta}{\delta z}\ln Z$$

$$= z\frac{\delta}{\delta z}[-\sum_k \ln\left(1 - e^{-\beta\hat{\epsilon}_k}\right)$$

$$= \sum_k (-z)\frac{1}{1 - e^{-\beta\hat{\epsilon}_k}}\left(-e^{-\beta\hat{\epsilon}_k}\right)$$

$$= \sum_k \frac{ze^{-\beta\hat{\epsilon}_k}}{1 - e^{-\beta\hat{\epsilon}_k}}, z = e^{\beta\mu}$$

$$= \sum_k \frac{e^{-\beta(\hat{\epsilon}_k - \mu)}}{1 - e^{-\beta(\hat{\epsilon}_k - \mu)}}$$

$$= \sum_k \frac{e^{-\beta(\hat{\epsilon}_k - \mu)} \times 1}{e^{-\beta(\hat{\epsilon}_k - \mu)}[e^{\beta(\hat{\epsilon}_k - \mu)} - 1]}$$

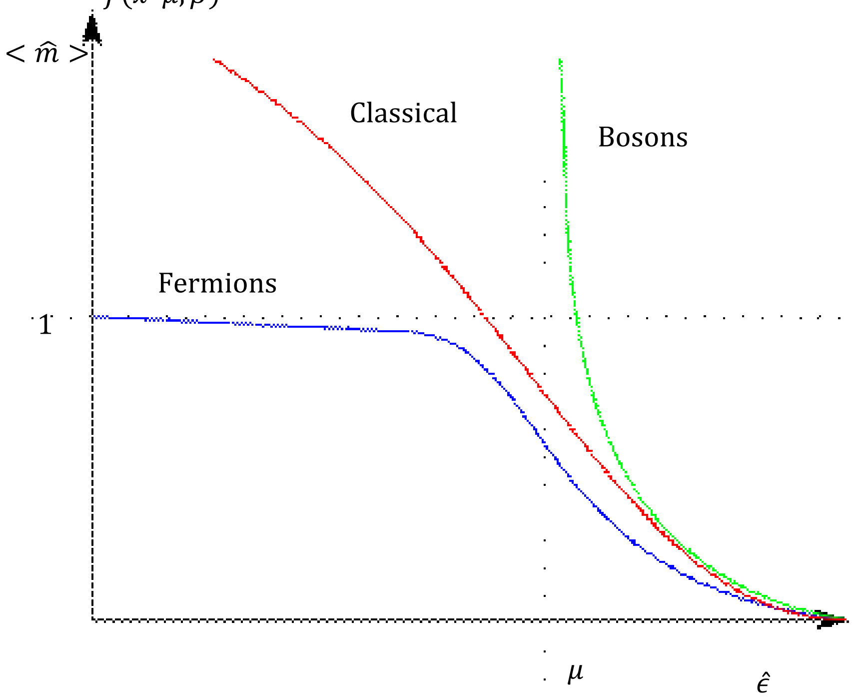
$$= \sum_k \frac{1}{-1 + e^{\beta(\hat{\epsilon}_k - \mu)}}$$

$$\boxed{< \hat{m}_k > = \frac{1}{-1 + e^{\beta(\hat{\epsilon}_k - \mu)}}}$$

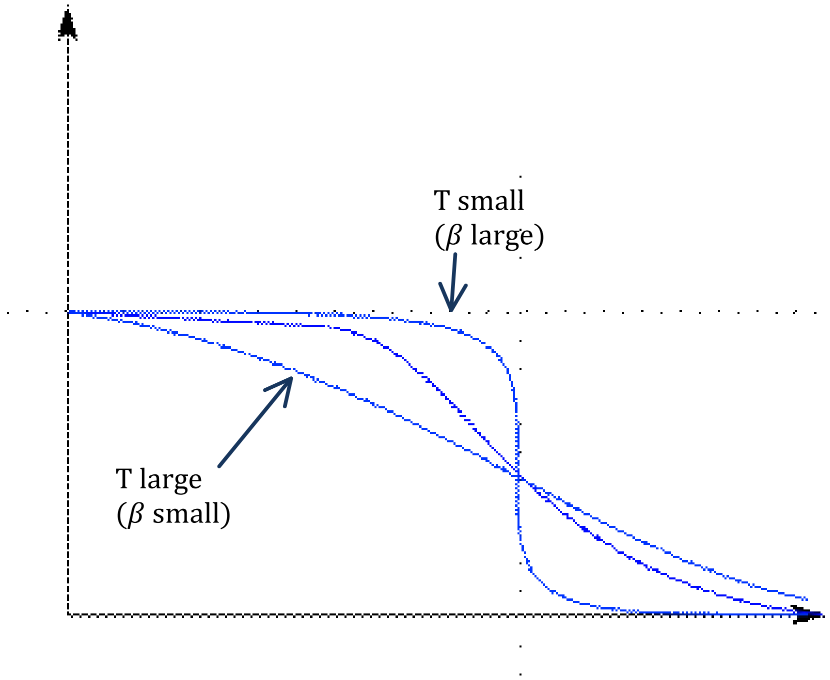
Bose-Einstein distribution

$$< \hat{m}_k > \ = \frac{1}{e^{\beta(\hat{\epsilon}_k - \mu)} + s}$$

s=-1 for bosons (Bose-Einstein)
s=+1 for fermions (Fermi-Dirac)
s=0 for semi-classical (Maxwell-Boltzmann)
$f(x \ \mu, \beta)$

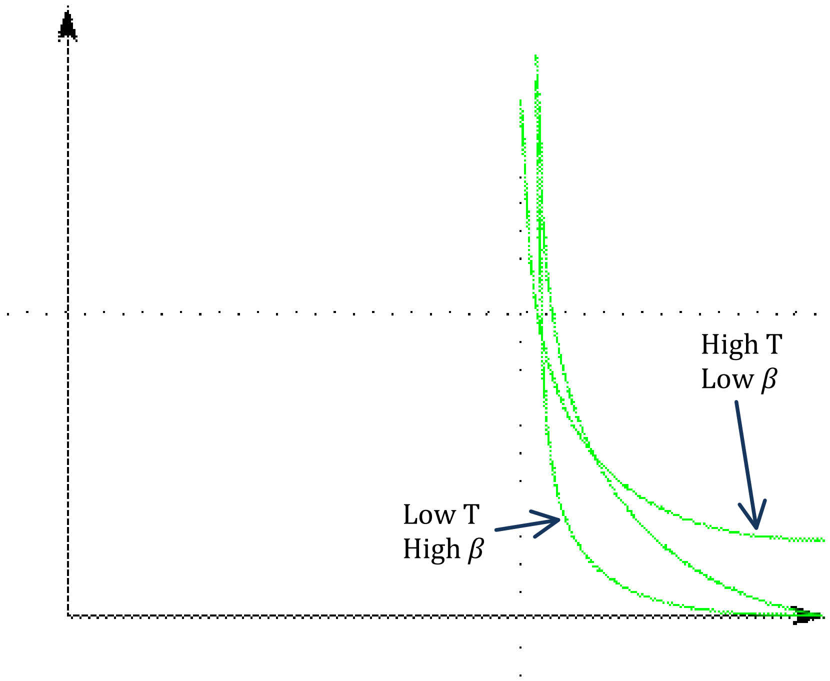


Fixed change β
Fermions
μ



Bosons

ϵ



To sumarise
1. Derived Quantum Gas distributions (with G. Canonical
2. At large T, or at large energy, all distributions agree with classical mechanics in physics
3. At low-T, huge differences appear

    Bosons, fermions completely different

Homework

15 December 2011
09:08

Relativistic GAS

$H_i = cP_i$

i=1,...,N

N=number particles

$P_i \equiv \sqrt{P_{i1}^2 + P_{i2}^2 + P_{i3}^2}$

$H = \sum_{i=1}^{N} H_i$

Set h=1, compute partition function

$$Z_N = \frac{1}{N!}\int d^{3N}q\ d^{3N}p\ e^{-\beta H}$$
$$= \frac{1}{N!}\int d^3p_1 d^3q_1 e^{-\beta H_1} d^3p_2 d^3q_2 e^{-\beta H_2} d^3p_3 d^3q_3 \ldots$$
$$= \frac{1}{N!}\left[\int d^3p_1 d^3q_1 e^{-\beta H_1}\right]^N$$

Polar coordinates in $\bar{P}_1$

$d^3p_1 = d\theta d\phi dp_1 p_1^2 \sin\theta$

$\int d\theta d\phi \sin\theta = 4\pi$

H_i does not depend on coordinates $\bar{q}_1$

$\int d^3q_1 = V$

$$= \frac{1}{N!}(4\pi V)^N \left[\int dp\ p^2 e^{-\beta cp}\right]^N$$

$$\int_0^\infty dx\ x^2 e^{-gx} = \frac{2}{g^3}$$

$$= \frac{(4\pi V)^N}{N!}\left[\frac{2}{\beta^3 c^3}\right]^N$$
$$= \frac{(4\pi V)^N}{N!}\left[\frac{2k^3T^3}{c^3}\right]^N$$
$$= \frac{(c_0 V T^3)^N}{N!}$$

$$F = -kT \ln Z_N$$
$$= -kT \ln \frac{(c_0 V T^3)^N}{N!}$$
$$= -NkT \ln c_0 V T^3 + kT \ln N!$$
$$= -NkT \ln(c_0 V T^3) + kT\ (N \ln N - N)$$
$$= -NkT[\ln(c_0 V T^3) - \ln N + 1]$$

$1 = \ln e$

$$= -NkT\left[\ln \frac{c_0 V T e}{N}\right]$$

Since c_0 is arbitrary constant, $c_0 \to ec_0$

$$= -NkT \ln \frac{c_0 V T^3}{N}$$

Pressure

$$P = -\frac{\delta F}{\delta V}$$
$$= -\frac{\delta}{\delta V}\left[-NkT \ln \frac{c_0 V T^3}{N}\right]$$
$$= \frac{NkT}{V}$$

Entropy

$S = -\frac{\delta F}{\delta T}$

$= -\frac{\delta}{\delta T}\left[-NkT \ln \frac{c_0 V T^3}{N}\right]$

$= -\frac{F}{T} + 3Nk$

Internal energy

$U = F + Ts$

$= F + T\left(-\frac{F}{T} + 3Nk\right)$

$= 3NkT$

Specific Heat

$C_v = \frac{\delta U}{\delta T} = 3Nk$

This result does NOT agree with the naïve version of the equipartition theorem because it does NOT depend quadratically on P!

However: it does agree with the generalized equipartition theorem

$< x_i \frac{\delta H}{\delta x_j} > \; = kT\delta_{ij}$

$H = H(x_i)$

$\leftarrow p, q, e, \ldots$

Then

$x \frac{\delta H}{\delta x} = 2x^2 = 2H$

$< H > \; = \frac{1}{2} < x \frac{\delta H}{\delta x} > \; = \frac{1}{2} kT$

Sum over all phase-space variable that enter quadratically

In exercise

$H = cp$

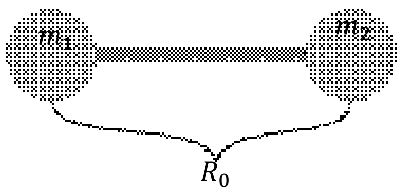
$P \frac{\delta}{\delta P} H = cp = H$

Exercise

N molecules of same diatomic gas with

$H_i = \frac{1}{2m} P_i^2 + \frac{1}{2I} l_i^2 + \frac{1}{2\mu} P_i^2 + \frac{1}{2}\mu\omega^2 (R - R_0)^2$

$\frac{1}{2\mu} P_i^2 + \frac{1}{2}\mu\omega^2 (R - R_0)^2$ →vibrational modes



Phase-space for each molecule

$\{\bar{P}, \bar{Q}, \bar{l}, \theta, \phi, R, \phi\}$

12 = 3 3 2 1 1 1 1

$\{p_1, q_1, P_2, q_2\}$

$\int d^{3N} Q$

$d^{3N}Q \; d^{3N}P \; d^{2N}L \; d^N\theta d^N\phi d^N R \; d^p \times (\sin\theta \, R^2)^N$

Compute Z_N

$T^{\frac{3}{2}} \, T^3 \, T^{\frac{5}{3}}$