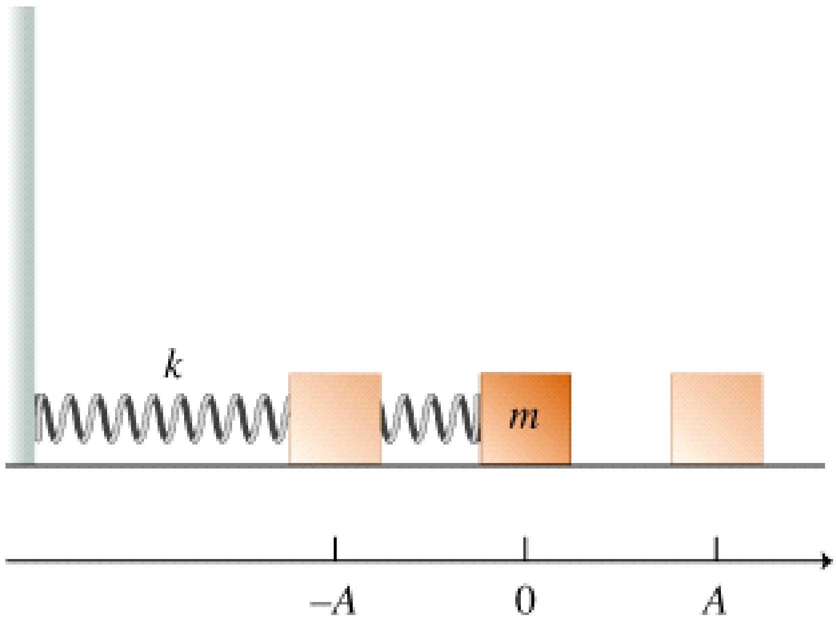
Simple harmonic motion

02 February 2011
10:10



$F \propto -A$

Force opposes the displacement in A

We assume the spring is linear

$F = -kA$

k is the spring constant. Sometimes called stiffness constant

Newton's law

$$M\frac{\delta^2 x(t)}{\delta t^2} = -kx(t)$$

$$\boxed{\frac{\delta^2 x(t)}{\delta t^2} = -\frac{k}{M}x(t)}$$

Frictionless (or undamped) SHM

Friction/damping observed to be proportional to velocity of oscillator

<u>Assume</u> damping $\propto v = \frac{\delta x}{\delta t}$ (velocity)

$$F = M\frac{\delta^2 x}{\delta t^2} = -kx(t) - \gamma\frac{\delta x(t)}{\delta t}$$

2nd order differential eq

Dividing by M and rearranging

$$\frac{d^2 x(t)}{dt^2} + \left(\frac{k}{M}\right)x(t) + \left(\frac{\gamma}{M}\right)\frac{dx(t)}{dt} = 0$$

Suppose

$x_1(t)$ is a solution

$x_2(t)$ is a second solution

$A\,x_1(t) + B\,x_2(t) = 0$ *will also be a solution*

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 + 2k\frac{dx_1}{dt} = 0$$
$$\frac{d^2 x}{dt^2} + \omega^2 x_2 + 2k\frac{dx_2}{dt} = 0$$

$$Eq(1) * A + Eq(2) * B = 0$$

$$\frac{d^2}{dt^2}(Ax_1 + Bx_2) + \omega^2(Ax_1 + Bx_2)$$

Linear superposition of 2 solutions is also a solution

$\ddot{x}(t) + 2k\dot{x}(t) + x(t) = 0$

2 independent solutions

To solve this, we must first assume an exponential form for

$x(t) \propto e^{\lambda t}$

(trial)

Plug this into equation

$\ddot{x}(t) + 2k\dot{x}(t) + \omega^2 x(t) = 0$

$x(t) \propto e^{\lambda t}$

$(\lambda^2 + 2k\lambda + \omega^2)e^{\lambda t} = 0$

$\therefore \lambda^2 + 2k\lambda + \omega^2 = 0$

$$\Rightarrow \lambda = \frac{-2k \pm \sqrt{4k^2 - 4\omega}}{2} = -k \pm \sqrt{k^2 - \omega}$$

$\ddot{x}(t) + 2k\dot{x}(t) + \omega^2 x(t) = 0$

$\therefore$ two independent solutions

$x_1(t) = \exp\left(-kt + \sqrt{k^2 - \omega^2} * t\right)$

$x_2(t) = \exp\left(-kt - \sqrt{k^2 - \omega^2} * t\right)$

Case I $\omega > k$: undamped

$x_1(t) = \exp(-kt)\exp(\mathrm{i}\sqrt{\omega^2 - k^2} * t)$

$x_1(t) = \exp(-kt)\exp\left(-\mathrm{i}\sqrt{\omega^2 - k^2} * t\right)$

Simplest solution, K=0

Damped oscillation (SHM)

04 February 2011
10:08

$\ddot{x}(t) + 2k\dot{x}(t) + \omega^2 x(t) = 0$
$x(t) \propto e^{\lambda t}$
Solve for λ, by plugging in
$\lambda = -K \pm \sqrt{k^2 - \omega^2}$
=>
Two linearly independent solutions

$\exp\left(-Kt + \sqrt{k^2 - \omega^2}t\right)$
$\exp\left(-Kt - \sqrt{k^2 - \omega^2}t\right)$

Three different cases

i. $\omega > K \rightarrow$ underdamped
ii. $\omega = k \rightarrow$ critical damping
iii. $\omega < K \rightarrow$ overdamped

Case (i) underdamped

Two independent solutions can be expressed as

$\exp\left(-Kt + \sqrt{k^2 - \omega^2}t\right)$
$\exp\left(-Kt - \sqrt{k^2 - \omega^2}t\right)$

Simplest solution, K=0
Two solutions are $\exp(\pm i\omega t)$
Most general solution is a linear combination
$x(t) = A_+ e^{+i\omega t} + A_- e^{-i\omega t}$
$= A_+[\cos(\omega t) + i\sin(\omega t)] + A_-[\cos(\omega t) - i\sin(\omega t)]$
$x(t) = (A_+ + A_-)\cos(\omega t) + i(A_+ - A_-)\sin(\omega t)$
To make x(t) real, we need to choose
$A_+ = (A_-)^* \rightarrow complex\ conjugate$
Can always write
$x(t) = A\cos(\omega t) + B\sin(\omega t)$
Where A and B are some real numbers
$x(t) = C\sin(\omega t + \delta)$

- Exercise: show that
- $C = \sqrt{A^2 + B^2}$
- $\delta = \tan^{-1}\left(\frac{A}{B}\right)$

$\delta = phase\ shift$
$\omega = angular\ frequency\ of\ oscillation$
δ shifts the figure to the right or left
$\delta = 0: x(t) = C\sin(\omega t)$
$\delta = \frac{\pi}{2}: x(t) = C\cos(\omega t)$
$C = amplitude\ of\ oscillation$
$T = time\ period\ of\ oscillation = \frac{2\pi}{\omega}$
$\omega \rightarrow units\ radians/sec$

Case with damping $k \neq 0$
Underdamped $K < \omega$
$\rightarrow$ two independent solutions
$\exp\left(-Kt \pm i\sqrt{(\omega^2 - k^2)}t\right)$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-kt} \qquad\qquad e^{\pm i\sqrt{\omega^2-k^2}t}$$
$$\downarrow \qquad\qquad \downarrow$$
decreases with time exponentially $(K > 0)$ *Oscillatory behaviour*

=Take linear combinations of the two solutions,
$x(t) = e^{-kt}(A\cos(\omega_D t) + \mathrm{B}\sin(\omega_\mathrm{D}\mathrm{t}) = \mathrm{C}\ \mathrm{e}^{-\mathrm{Kt}}\sin(\omega_D t + \delta)$

$\omega_D = \sqrt{\omega^2 - k^2} \rightarrow$ *frequency is smaller than undamped case*

Plot, $\delta = 0$
$x(t) = Ce^{-Kt}\sin\left(\sqrt{\omega^2 - k^2}t\right)$
$T = \dfrac{2\pi}{\sqrt{\omega^2 - k^2}}$
When $k \rightarrow \omega$, the time period diverges → system is said to be
$\boxed{critically\ damped}$

New solution emerges at critical damping
$x(t) = Ae^{-kt} + Bte^{-kt}$

Case 3: k=w critically damped oscillation
The harmonic oscillator equation becomes
$\ddot{x} + \omega^2 x + 2\omega^2\dot{x} = 0$
If we follow the same method as before and choose a trivial solution $\propto e^{at}$
Plugging into equation $\lambda^2 + 2\omega\lambda + \omega^2 = 0$
$(\lambda + \omega)^2 = 0 \Rightarrow \lambda = -\omega \rightarrow onle\ one\ root!$
Only one solution is of an exponential form $= e^{-\omega t}$
2nd solution can be guessed by looking at the general solutions with $\omega \neq k$
$\omega_0 = \sqrt{\omega^2 - k^2}\begin{cases} e^{-kt}\cos(\omega_0 t) \\ e^{-kt}\sin(\omega_0 t) \end{cases}$
As $k \rightarrow \omega$
$\omega_0 \rightarrow 0$
1st solution
$e^{-kt}\cos(\omega_0 t) \rightarrow e^{-kt}$
2nd solution
$e^{-kt}\cos(\omega_0 t) \rightarrow e^{-kt}(\omega_0 t)$

So 2 independent solutions for critically damped motion
$e^{-kt} and\ te^{-kt}$
Exercise: plug $x(t) = te^{-kt}\ in\ \ddot{x} + 2k\dot{x} + k^2 x = 0$
And show that it satisfies
General solution
$x(t) = Ae^{-kt} + Bte^{-kt}$
(A and B are two constants of integration)
-> non oscillatory motion
Suppose at t=0, the oscillator is at x=0
$x(t = 0) = A = 0$
Fixes one integration constant
$x(t) = Bte^{-ke}$

Case 3: overdamped w<k

$\ddot{x} + 2k\dot{x} + \omega^2 x = 0$
Plug in $x(t) \propto \exp(\lambda t)$
$\lambda^2 + 2k\lambda + \omega^2 = 0$
$\lambda = -k \pm \sqrt{k^2 - \omega^2}$

Two independent solutions
$e^{-kt}e^{+\sqrt{k^2-\omega^2}t}$
$e^{-kt}e^{-\sqrt{k^2-\omega^2}t}$
Real roots
Both solutions are exponentially decaying

Forced oscillations

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14:02

$\ddot{x} + 2k\dot{x} + \omega^2 x = F(t)$
F(t) is forcing function. In general it can be anything
Periodic forcing functions are "building blocks" for any kind of forcing functions (fourier analysis)
Therefore it suffices to look at F(t) that are periodic in time with some frequency
Take
$F(t) = F_0 \sin(\alpha t)$
Solve the motion
$\ddot{x}(t) + 2k\dot{x}(t) + \omega^2 x(t) = F_0 \sin(\alpha t)$
Inhomogenous differential equation
Suppose I find a solution
$x_p(t)$
"particular" solution
I can always add to $x_P(t)$, a function which we call $x_H(t)$
Homogenous solution
Add to $x_P(t)$, *some* $x_H(t)$
$x_P(t) + x_H(t)$ →*also satisfies the forced oscillator equation*
This solves the equation $\boxed{\ddot{x}_H + 2k\dot{x}_H + \omega^2 x_H = 0}$

$F_0\, sin(\alpha t)$ →In homogen*ous*
forcing function

We look for a particular solution
Resulting motion consists of a piece that is determined by natural frequency (ω) of system
+
a piece determined by forcing frequency (α)
$xp(t) = M\cos(\alpha t) + N\sin(\alpha t)$
Plug into equation
$(-M\alpha^2\cos(\alpha t) - N\alpha^2\sin(\alpha t)) + 2k(-M\alpha\sin(\alpha t) + N\alpha\cos(\alpha t)) + \omega^2(M\cos(\alpha t) + \sin(\alpha t))$
$= F_0\sin(\alpha t)$
$\Rightarrow \sin(\alpha t)\,(-N\alpha^2 - 2M\alpha k + \omega^2 N) + \cos(\alpha t)\,(-M\alpha + 2N\alpha k + M\omega^2) = F_0\sin(\alpha t)$

2 equations for m and n

1. $M\omega^2 + 2Nak + \omega^2 N = 0 \Rightarrow M = -\dfrac{2N\alpha k}{\omega^2 - \alpha^2}$
2. $-N\alpha^2 - 2M\alpha k + \omega^2 N = F_0$

$\Rightarrow M = -\dfrac{2N\alpha k}{\omega^2 - \alpha^2}$
Plug in M in equation 2

$$Exercise: \boxed{N = \frac{F_0(\omega^2 - \alpha^2)}{(\omega^2 - \alpha^2)^2 + 4k^2\alpha^2}}$$

$$M = -\frac{2kF_0\alpha}{(\omega^2 - \alpha^2)^2 + 4k^2\alpha^2}$$

$x_P(t) = M\cos(\alpha t) + N\sin(\alpha t)$
Rewrite this expression
$= \sqrt(M^2 + N^2)\left(\dfrac{M}{\sqrt{M^2 + N^2}}\cos(\alpha t) + \dfrac{N}{\sqrt{M^2 + N^2}}\sin(\alpha t)\right)$
$\dfrac{M^2}{M^2 + N^2} + \dfrac{N^2}{M^2 + N^2} = 1$

$x_p(T) = \sqrt{M^2 + N^2}(-\sin\delta\cos(\alpha t) + \cos\delta\sin(\alpha t)) = \sqrt{M^2 + N^2}(\alpha t - \delta)$

You'll get

$$\frac{F_0}{\sqrt{(\omega^2 - \alpha^2)^2 + 4k^2\alpha^2}} sin(\alpha t - \delta)$$

When the forcing frequency approaches the natural frequency, amplitude is at a maximum

Extreme of the function

$$\frac{\delta}{\delta\alpha}\left(\frac{F_0}{\sqrt{(\omega^2 - \alpha^2)^2 + 4k^2\alpha^2}}\right) = -\frac{1}{2}\frac{F_0}{\sqrt{(\omega^2 - \alpha^2)^2 + 4k^2\alpha^2}}(2(\omega^2 - \alpha)(-2\alpha) + 8k^{2\alpha} = 0$$

$\Rightarrow 8k^2\alpha - 4\alpha(\omega^2 - \alpha^2) = 0$

$2k^2 - (\omega^2 - \alpha^2) = 0$

$\boxed{\alpha^2 = \omega^2 - 2k}$

Amplitude has a peak when $\alpha = \sqrt{\omega^2 - 2k^2}$

In the absence of damping (k small) resonance occurs at $\alpha = \omega$

At the maximum, the size of the amplitude $(\alpha = \sqrt{\omega^2 - 2k^2})$

$$\frac{F_0}{2k\omega\sqrt{1 - \frac{k^2}{\omega^2}}}$$

When damping is small, $\frac{k}{\omega} \ll 1$

Amplitude=

$$\frac{F_0}{2k\omega\sqrt{1 - \frac{k^2}{\omega^2}}}$$

$$\cong \frac{F_0}{2k\omega} \rightarrow diverges\ as\ k \rightarrow 0$$

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09:16

Due to spherical symmetry by Gauss' law to gravity, acceleration due to gravity is due to the total mass inside a sphere of radius "r"

$$\frac{d^2r}{dt^2} = -\left(\frac{4}{3}G\pi\rho\right)r$$
$$\frac{d^2r}{dt^2} = -\omega^2 r$$

$$\omega = \sqrt{\frac{4}{3}G\pi\rho}$$

Time period of oscillation

$$T = \frac{2\pi}{\sqrt{\frac{4}{3}\pi G\rho}} = \frac{2\pi}{\sqrt{\frac{4}{2}\pi G\left(\frac{M_E}{4\pi R_E^3}\right)}} = \frac{2\pi\sqrt{R_E^3}}{\sqrt{GM_E}} \cong 83\ minutes$$

∴ solution: $r(t) = A\sin(\omega t) + B\cos(\omega t)$

At t=0

$r(t=0) = R_E$

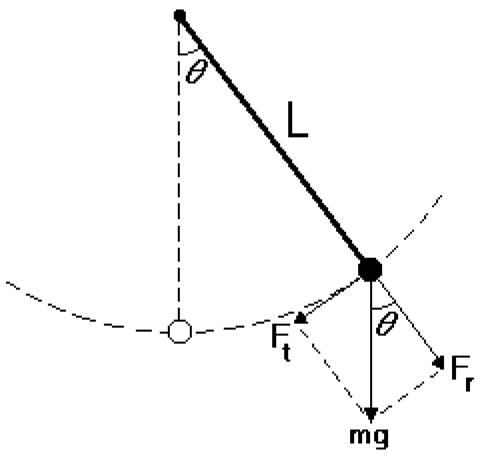
$\dot{r}(t=0) = 0$ At rest

$r(t=0) = B = R_E$

$\dot{r}(t=0) = A\omega = 0 \Rightarrow A = 0$

∴ full solution

$r(t) = R_E\cos(\omega t)$



for t<0

$\theta(t) = \theta_0\sin(\omega_0 t)$

$\omega_0 = natural\ frequency = \sqrt{\frac{g}{L}}$

For times t>0

Forcing function $F(t) = F_0\sin(\omega_F t)$

Forced motion for times t>0

Simple pendulum

$$\ddot{\theta} + \omega_0^2\theta(t) = 0$$

Equation of simple harmonic motion

Add forcing function
$\ddot{\theta} + \omega_0^2\theta(t) = F_0 \sin(\omega_F t)$

Full solution
$\theta(t) = \underset{unforced\ oscillation}{\theta_0 \sin(\omega_0 t)} + \underset{forced\ oscillation}{\theta_F(t)}$
Take
$\theta_F(t) = A\sin(\omega_F t) + B\cos(\omega_F t)$
Plug into equation to determine A and B
$\ddot{\theta} + \omega_0^2 = F_0 \sin(\omega_F t)$
$\Rightarrow -\omega_F^2(A\sin(\omega_F t) + B\cos(\omega_F t)) + \omega_0^2(A\sin(\omega_F t) + B\cos(\omega_F t)) = F_0 \sin(\omega_F t)$
$(-\omega_F^2 + \omega_0^2)(A\sin(\omega_F t) + B\cos(\omega_F t)) = F_0 \sin(\omega_F t)$
B=0
$A(-\omega_F^2 + \omega_0^2) = F_0$
$\Rightarrow A = \frac{F_0}{-\omega_F^2 + \omega_0^2}$
$\therefore \theta(t) = \theta_0 \sin(\omega_0 t) + \frac{F_0}{-\omega_F^2 + \omega_0^2}\sin(\omega_F t)$
Amplitude of forced oscillations
$= \left|\frac{F_0}{\omega_0^2 - \omega_F^2}\right|$

For simplicity, suppose
$\frac{F_0}{\omega_0^2 - \omega_F^2} = \theta_0$

$\theta(t) = \theta_0(\sin(\omega_0 t) + \sin(\omega_F t))$
$\theta_0\left(2\sin\left(\frac{\omega_0 + \omega_F}{2}t\right)\cos\left(\frac{\omega_0 - \omega_F}{2}t\right)\right)$
Assume $\omega_F \sim \omega_0$, not necessarily equal
$\frac{\omega_F + \omega_0}{2} \sim \omega_0$
To a good approximation
$\theta(t) = \theta_0\left[2\sin(\omega_0 t)\cos\left(\frac{\omega_F - \omega_0}{2}t\right)\right]$

The cosine modulates the amplitude
ω_F close to ω_0 implies that
$\cos\left(\frac{\omega_F - \omega_0}{2}t\right)$
Is a slowly varying function
$T_{slow} = \frac{2\pi}{\omega_F - \omega_0}$
Fast oscillations have time period
$\frac{2\pi}{\omega_0}$
"Beats"
Example: travelling waves
$A(t,x) = A_1 \sin(\omega_1 t - kx) + A_2 \sin(\omega_2 t - kx)$
ω_1 *and* ω_2 are frequencies, x=position

Superposition of simple harmonic oscillations

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14:25

(1) same phase, different frequency

$x_1 = A\sin(\omega_1 t + \phi)$

$x_2 = A\sin(\omega_2 t + \phi)$

Superposition of $x_1(t)$ *and* $x_2(t)$

$x_1 + x_2 = A(\sin(\omega_1 t + \phi) + \sin(\omega_2 t + \phi)$

Saw just now-> leads to beats

(2)same frequency, different phases

$x_1 = A_1 \sin(\omega t + \phi_1)$

$x_2 = A_2 \sin(\omega t + \phi_2)$

$x_1(t) + x_2(t)$ also solves the simple harmonic equation $\ddot{x} + \omega^2 x = 0$

$x_1(t) + x_2(t)$

$= A_1(\sin(\omega t)\cos\phi_1 + \cos(\omega t)\sin\phi_1) + A_2(\sin(\omega t)\cos\phi_2 + \cos(\omega t)\sin\phi_2)$

Collecting terms proportional to sin and cos,

$x_1(t) + x_2(t) = \sin(\omega t)\,(A_1\, cos\,\phi_1 + A_2\cos\phi_2) + cos(\omega t)\,(A_1\, sin\,\phi_1 + A_2\, sin\,\phi_2)$

Recall the trick

$A\sin(\omega t) + B\cos(\omega t)$

$= \sqrt{A^2+B^2}(\frac{A}{\sqrt{A^2+B^2}}\sin(\omega t) + \frac{B}{\sqrt{A^2+B^2}}\cos(\omega t) = \sqrt{A^2+B^2}\sin(\omega t + \delta)$

Where

$\delta = \tan^{-1}\left(\frac{B}{A}\right)$

Ampltude

$\sqrt{(A_1\, cos\,\phi_1 + A_2\cos\phi_2)^2 + (A_1\, sin\,\phi_1 + A_2\, sin\,\phi_2)^2}$

$= \sqrt{A_1^2 + A_2^2 + 2A_1A_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2\,)}$

$Amplitude = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}$

If amplitudes $A_1 = A_2$

$= A_1\sqrt{2(1+\cos(\phi_1 - \phi_2)\,)}$

$\sqrt{2}A\sqrt{2\cos^2\frac{\phi_1 - \phi_2}{2}}$

$2A_1\cos\frac{\phi_1 - \phi_2}{2}$

$(x_1 + x_2) = 2A_1\cos\frac{\phi_1 - \phi_2}{2} \times \sin(\omega t + \delta)$

We learn

(a) if $\phi_1 = \phi_2$ then the functions are in phase

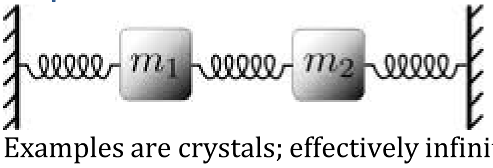
(b) if $\phi_1 - \phi_2 = \pi$, oscillations are out of phase; they cancel each other; destructive interference

Phase shift exercise

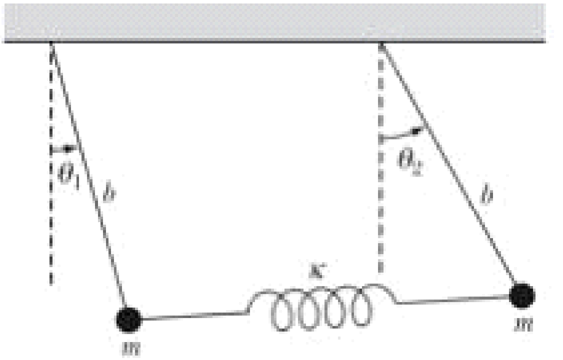
$\delta = \frac{\phi_1 + \phi_2}{2}$

If we assume that $A_1 = A_2$

Coupled oscillators+ normal modes



Examples are crystals; effectively infinite grid of coupled oscillators



$x_+(t) = A_+ \sin(\Omega_+ t) \rightarrow \Omega_+ = \omega_p \Rightarrow$ slow mode

$x_-(t) = A_- \sin(\Omega_- t) \rightarrow \Omega_- = \sqrt{\omega_p^2 + 2\omega_s^2} \Rightarrow$ fast $mode$

Most general solutions is a linear combination of $x_+ \& x_-$

$A_+ \sin(\Omega_+ t) + A_- \sin(\Omega_- t)$

Analytical mechanics

03 March 2011
13:06

Lagrangian Mechanics
Lagrange
Hamiltonian Mechanics
Hamilton

Equivalent to Newton's laws-> about forces and acceleration->can be messy
Forces are not the central objects-> the mechanics is determined by scalar quantities

<u>Lagrangians</u>

Consider a particle of mass "m" moving in the influence of a potential V(x)
From newton,

$$m\frac{d^2x}{dt^2}=-\frac{dV(x)}{dx}$$

$$\frac{d}{dt}(m\dot{x})=-\frac{dV(x)}{dx}$$

Force =derivative of momentum

$$\frac{d}{dt}\left(\frac{\delta}{\delta\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right)=-\frac{dV(x)}{dx}$$

T=kinetic energy= $\frac{1}{2}m\dot{x}^2$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot{x}}\right)=-\frac{\delta V}{\delta x}$$

For every 'mechanical' system there exists a Lagrangian L

L=T-V

$$L=\frac{1}{2}m\dot{x}-V(x)$$

$$L=L(x,\dot{x})$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{x}}\right)=\frac{\delta L}{\delta x}$$

$$\frac{d}{dt}\left(\frac{\delta L(x,\dot{x})}{\delta\dot{x}}\right)=\frac{\delta L(x,\dot{x})}{\delta x}$$

Once L is known, equations of motion follow, no need to talk about forces
In general

$$L=L(q_1,q_2,q_3\dots;\dot{q}_1,\dot{q}_2,\dot{q}_3\dots)$$

q_1,q_2 need not be cartesian coordinates

$q_1,q_2\rightarrow generalized\ coordinates$

Point particle in a potential $L=\frac{1}{2}m\dot{x}^2-V(x)$

Calculate

$$\frac{\delta L}{\delta\dot{x}}=m\dot{x}=p_x\rightarrow canonical\ or\ generalized\ momentum$$

$$\frac{\delta L}{\delta x}=-V'(x)=F_x$$

$$\frac{d}{dt}(p_x)=F_x$$

<u>Apply to pendulum</u>

Kinetic energy of the pendulum

$$T=\frac{1}{2}m\left(L\dot{\theta}\right)^2=\frac{1}{2}mL^2\theta^2$$

$$V(\theta)=mgL(1-\cos\theta)$$

$$L=T-V=\frac{1}{2}mL^2\dot{\theta}^2-mgl(1-\cos\theta)$$

Generalized coordinate θ

$$p_\theta=\frac{\delta L}{\delta\dot{\theta}}=Generalixed\ momentum=mL^2\dot{\theta}=angular\ momentum$$

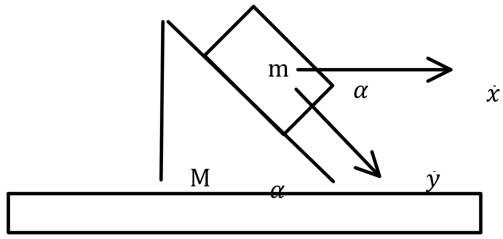
$$F_\theta=\frac{\delta L}{\delta}\theta=-mgl\sin\theta=generalized\ force$$

$$\frac{d}{dt}(P_\theta)=F_\theta$$

$$\frac{d}{dt}(ml^2\dot{\theta}) = -mgl\sin\theta$$

$$\Rightarrow ml^{2\ddot{\theta}} = -mgl\sin\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta \rightarrow equation\ \ of\ motion$$



Mass on an inclined plane on a frictionless surface

Inclined plane: $\frac{1}{2}M\dot{x}^2$

Block of mass m: Velocity is a vector sum of 2 components

$$(\dot{x} + \dot{y})^2$$

$$= \dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha$$

Kinetic energy for

'm'; $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha)$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha)$$

$$L = L(x, y, \dot{x}, \dot{y}) = T - V$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha) + mgy\sin\alpha$$

Note:V is independent of x

Generalized momenta:

$$\frac{\delta L}{\delta \dot{x}} = p_x = M\dot{x} + m\dot{x} + m\dot{y}\cos\alpha$$

$$\frac{\delta L}{\delta \dot{y}} = p_y = m\dot{y} + m\dot{x}\cos\alpha$$

Generalized forces

$$\frac{\delta L}{\delta x} = F_x = 0$$

$$\frac{\delta L}{\delta y} = F_y = mg\sin\alpha$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta y}\right) = \frac{\delta L}{\delta y}$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta x}\right) = \frac{\delta L}{\delta x}$$

(1) $M\ddot{x} + m\ddot{x} + m\ddot{y}\cos\alpha = 0$

$$\ddot{x}(M + m) = -m\ddot{y}\cos\alpha$$

$$\ddot{x} = -\frac{m\ddot{y}\cos\alpha}{M + m}$$

(2)$m\ddot{y} + m\ddot{x}\cos\alpha = mg\sin\alpha$

2 equations for $\ddot{x} and\ \ddot{y}$

$$\ddot{x} = -\frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

$$\ddot{y} = \frac{(M + m)g\sin\alpha}{M + m\sin^2\alpha}$$

<u>Why does this work?</u>

Hamlton's action principle

For every system there exists quantity called action

$$S = \int_{t_i}^{t_f} dt\ L(q_1, q_2, q_3 \ldots; \dot{q}_1, \dot{q}_2, \dot{q}_3 \ldots)$$

The action is 'extremized' only when q_1, q_2, q_3 obey the lagrange equations of moton

Lease action principle

10 March 2011
09:21

Action
$S = \int_{t_i}^{t_f} dt\ L\big(q(t), \dot{q}(t)\big)$
Subject to boundary contitions
$q(t_i) = q_i$
$q(t_f) = q_f$
Consider motion of one particle in one direction
Trajectory x(t)
Small change in trajectory
$x(t) + \epsilon g(t)(\epsilon \ll 1)$
$f(y+\epsilon) - f(y) = \delta f = change\ in\ f = \frac{df}{dy} * \epsilon$
If $\frac{df}{dy} = 0$
Then $\delta f = 0 \Rightarrow$ we are at a maximum or minimum
Let us evaluate the change in the action
$S = \int_{t_i}^{t_f} dt\ L\big(x(t) + \epsilon g(t); \dot{x}(t) + \epsilon \dot{g}(t)\big)$
What is the change or variation

$\delta S = \int_{t_i}^{t_f} dt\ L\big(x(t) + \epsilon g(t); \dot{x}(t) + \epsilon \dot{g}(t)\big) - \int_{t_i}^{t_f} dt\ L(x, \dot{x})$
$L(x + \epsilon g, \dot{x} + \epsilon \dot{g})$
Taylor series for small ϵ
$\cong L(x, \dot{x}) + \epsilon g(t) \frac{\delta L}{\delta x} + \epsilon \dot{g}(t) \frac{\delta^2 L}{\delta x^2}$

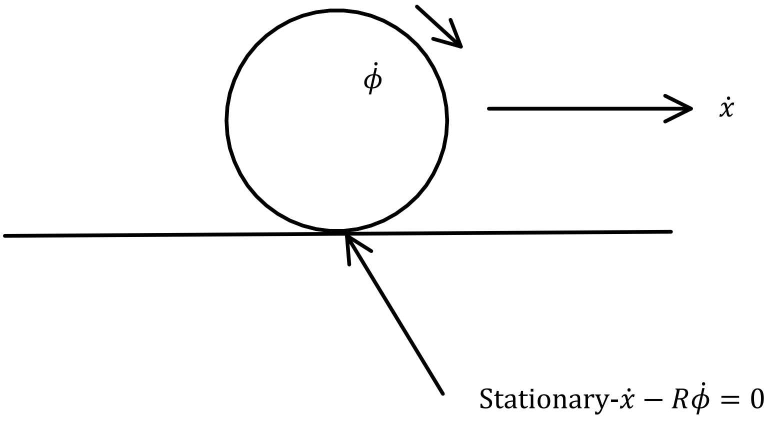
$\delta S = \int_{t_i}^{t_f} dt\ \left[\epsilon g(t) \frac{\delta L}{\delta x} + \epsilon \dot{g}(t) \frac{\delta^2 L}{\delta x^2}\right]$
$\dot{g}(t) \frac{\delta^2 L}{\delta x^2} = \frac{d}{dt}\left(g(t) \frac{\delta^2 L}{\delta x^2}\right) - g(t) \frac{d}{dt}\left(\frac{\delta^2 L}{\delta x^2}\right)$
Plug into δS:
$\int_{t_i}^{t_f} dt\ \left[\epsilon g(t) \frac{\delta L}{\delta x} - \epsilon \dot{g}(t) \frac{\delta^2 L}{\delta x^2} + \epsilon \frac{d}{dt}\left(g(t) \frac{\delta^2 L}{\delta x^2}\right)\right]$
The third term
$\epsilon \int_{t_i}^{t_f} dt\ \left[\frac{d}{dt}\left(g(t) \frac{\delta^2 L}{\delta x^2}\right)\right] = \epsilon \left[g(t_f) \frac{\delta^2 L}{\delta x^2}\Big|_{t_f} - g(t_i) \frac{\delta^2 L}{\delta x^2}\Big|_{t_i}\right] = 0$

$\delta S = \int_{t_i}^{t_f} dt\ \left[\epsilon g(t) \frac{\delta L}{\delta x} - \epsilon \dot{g}(t) \frac{\delta^2 L}{\delta x^2}\right]$
$\delta S = \epsilon \int_{t_i}^{t_f} dt\ g(t) \left[\frac{\delta L}{\delta x} - \frac{d}{dt}\left(\frac{\delta^2 L}{\delta x^2}\right)\right]$
If $\delta S = 0$
$\Rightarrow \frac{\delta L}{\delta x} = \frac{d}{dt}\left(\frac{\delta^2 L}{\delta x^2}\right) \rightarrow Lagrange\ equations\ of\ motion$

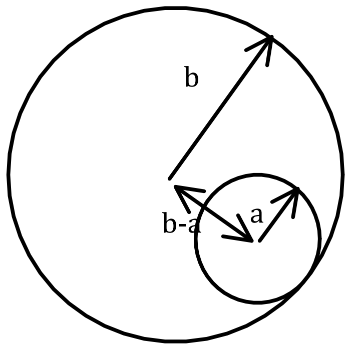
Cylinder rolling without slipping

17 March 2011
13:08

Constrained motion



$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{4}MR^2\dot{\phi}^2 = \frac{3}{4}MR^2\dot{\phi}^2$
$L = T \quad (v = 0)$
$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\phi}}\right) = \frac{\delta L}{\delta \phi} = 0$
$\frac{3}{4}MR^2 2\ddot{\phi} = 0$
$\ddot{\phi} = 0$
$\phi = \omega t + c$



Small cylinder rolling without slipping inside bigger hollow cylinder

$L = T - V$
$T = \frac{1}{2}M(b-a)^2\dot{\theta}^2 \rightarrow Rotational\ KE\ for\ rotations\ around\ axis\ of\ big\ cylider$
$+\frac{1}{2}I\dot{\phi}^2$
$I = \frac{1}{2}Ma^2$
Constraint
$\frac{d\theta}{dt}(b-a) = \frac{d\phi}{dt}a$
Plug in for $\dot{\phi}$ in terms of$\dot{\theta}$
$\boxed{T = \frac{3}{4}M(b-a)^2\dot{\theta}^2}$
$\boxed{V = Mg(b-a)(1-\cos\theta)}$
$L = T - V = \frac{3}{4}M(b-a)^2\dot{\theta}^2 - Mg(b-a)(1-\cos\theta)$
Equations of motion
Generalized momentum,
$p_\theta = \frac{\delta L}{\delta \dot{\theta}} = \frac{3}{2}M(b-a)^2\dot{\theta}$
Generalized force

$$\frac{d}{dt}(p_\theta) = \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\theta}}\right) = \frac{3}{2}M(b-a)^2\ddot{\theta} = \frac{\delta L}{\delta \theta} = -Mg(b-a)\sin\theta$$

Equation of motion

$$\frac{3}{2}\boxed{M}(b-a)^2\ddot{\theta} = -\boxed{M}g(b-a)\sin\theta$$

For simple pendulum $\ddot{\theta} = -\frac{g}{L}\sin\theta$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{2}{3}\frac{g}{b-a}\sin\theta}$$

For small θ

$$\ddot{\theta} = -\frac{2}{3}\frac{g}{b-a}\theta \rightarrow (used \sin\theta \sim \theta\ for\ smal\ theta$$

Oscillations,

$$\omega = \sqrt{\frac{2}{3}\frac{g}{b-a}}$$

$$\frac{d^2x}{dt^2} = -nx$$

$$\omega = \sqrt{n}$$

$$\omega = \sqrt{\frac{2g}{3(b-a)}}$$

$$\rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3(b-a)}{2g}}$$

(if b=a when T->0

Motion of a particle in 2 dimensions

Potential energy is rotationally symmetric

$V = V(x,y)$ such that it is symmetric under rotations in the x-y plane

Polar coordinates natural for rotationally invariant/symmetric systems

Central potential in two dimensions

24 March 2011
09:05

V depends only on r
$V(r) \Rightarrow function\ of\ |\bar{r}|$
Choose to work in polar coordinates → $trade\ (x, y) for\ (r, \phi)$
Lagrangian $L = T - V$
$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$
Convert to polar coordinates
$x = r\cos\phi$
$\dot{x} = \dot{r}\cos\phi - (\sin\phi)r\dot{\phi}$
$y = r\sin\phi$
$\dot{y} = \dot{r}\sin\phi + (\cos\phi)\dot{\phi}r$
$T = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2)$
$= \frac{1}{2}m\big(\dot{r}^2\cos^2\phi + 2\sin^2\phi - 2r\dot{r}\cos\phi\sin\phi\,\dot{\phi} + \dot{r}^2\sin^2\phi + \cos^2\phi\,\dot{\phi}^2r^2$
$+ 2r\dot{r}\cos\phi\sin\phi\,\dot{\phi}\big)$
$T = \frac{1}{2}m\big(\dot{r}^2 + r^2\dot{\phi}^2\big) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2$
$\frac{1}{2}m\dot{r}^2 \Rightarrow ke\ of\ radial\ motion$
$\frac{1}{2}mr^2\dot{\phi}^2 \Rightarrow ke\ of\ angular\ motion$
$\therefore L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - V(r)$
What are the equations of motion?
Generalized momentum

$P_r = \frac{\delta L}{\delta \dot{r}} = m\dot{r};$

$P_\phi = \frac{\delta L}{\delta \dot{\phi}} = mr^2\dot{\phi} \Rightarrow angular\ momentum$

Generalized form

$\frac{dP_r}{dt} = \frac{\delta L}{\delta r} = mr\dot{\phi}^2 - V'(r)$

$\Rightarrow m\ddot{r} = mr\dot{\phi}^2 - V'(r)$

Radial component of force

$mr\dot{\phi}^2$ ⇒centripetal force

$V'(r)$ ⇒ force from the potential

$\frac{dP_\phi}{dt} = \frac{\delta L}{\delta \phi} = 0$

P_ϕ is a constant in time-> conservation of angular momentum!

$P_\phi = mr^2\dot{\phi} = constant$
Useful to express
$\dot{\phi} = \frac{P_\phi}{mr^2}$
$m\ddot{r} = mr\left(\frac{P_\phi^2}{m^2r^4}\right) - V'(r)$
$m\frac{d^2r}{dt^2} = \frac{P_\phi^2}{mr^3} - \frac{dV(r)}{dr}$
Suppose we define a new potential

$$\tilde{V}\,(r) = \frac{P_\phi^2}{2mr^2} + V(r)$$
$$P_\phi = constant$$

The lagrangian associated to such a potential

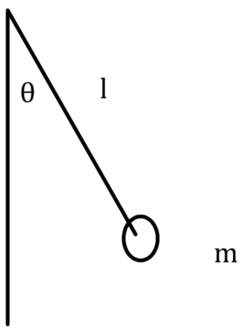
$$L = \frac{1}{2}m\dot{r}^2 - \tilde{V}(r)j$$

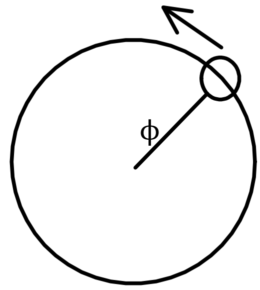
Eqn motion

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{r}}\right) = \frac{\delta L}{\delta r}$$

$$\Rightarrow m\ddot{r} = -\tilde{V}'(r) = \frac{P_\phi^2}{mr^3} - V'(r)$$

Pendulum rotating on an axis





Viewed from above

$$L = T - V$$
$$V = mgl(1 - \cos\theta)$$
$$T = \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}m\dot{\phi}^2L^2\sin^2\theta$$

Generalized momenta

$$\frac{\delta L}{\delta \dot{\theta}} = P_\theta = mL^2\dot{\theta} \Rightarrow angular\ momentum\ associate\ to\ \theta\ rotations$$

$$\frac{\delta L}{\delta \dot{\phi}} = P_\phi = mL^2\dot{\phi}\sin^2\theta \Rightarrow angular\ momentum\ associated\ to\ \phi$$

P_ϕ is conserved

Symmetry under rotation in ϕ- direction

$$L = \frac{1}{2}m\dot{\phi}^2l^2\sin^2\theta + \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$F_\theta = ml^2\ddot{\theta} = \frac{\delta L}{\delta\theta} = \frac{1}{2}m\dot{\phi}^2l^2(2\sin\theta \times \cos\theta) - mgl\sin\theta$$

$$\boxed{ml^2\ddot{\theta} = m\dot{\phi}^2l^2\sin\theta\cos\theta - mgl\sin\theta}$$

$$F_\phi = \frac{dP_\phi}{dt} = \frac{d}{dt}(ml^2\dot{\phi}\sin^2\theta = \frac{\delta L}{\delta\phi} = 0$$

$$\therefore \boxed{ml^2\dot{\phi}\sin^2\theta = constant}$$

P_ϕ is conserved

Since P_ϕ is constant, $ml^2\dot{\phi}\sin^2\theta = P_\phi = constant$

$$\therefore \boxed{\dot{\phi} = \frac{P_\phi}{ml^2\sin^2\theta}}$$

Plug into equation of motion for θ

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \phi^2\sin\theta\cos\theta$$
$$= -\frac{g}{l}\sin\theta + \frac{p_\phi^2}{m^2l^4\sin^4\theta}\sin\theta\cos\theta$$

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{P_\phi^2}{2m^2l^2\sin^3\theta}\cos\theta}$$

Depends only on θ

Define a new effective potential $\tilde{V}(\theta)$

$$\tilde{V}(\theta) = mgl(1-\cos\theta) + \frac{P_\phi^2}{m^2l^4\sin^2\theta}$$

Lagrangian associated to this potential

$$L = 12\, ml^2\dot{\theta}^2 - \tilde{V}(\theta)$$

Yields exactly the same equation of motion for θ

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\theta}}\right) = \frac{\delta L}{\delta\theta}$$

$$ml^2\ddot{\theta} \quad = -mgl\sin\theta - \frac{p_\phi^2}{m^2l^2}\left(\frac{\cos\theta}{\sin^2\theta}\right)$$

$$P_\phi = ml^2\sin^2\theta\,\dot{\phi} \Rightarrow ml^2\sin^2\theta\,\omega$$

$$\tilde{V}(\theta) = mgl(1-\cos\theta) + \frac{1}{2}\frac{(ml^2\sin^2\theta\,\omega)^2}{ml^2\sin^2\theta}$$

$$\tilde{V}(\theta) = mgl(1-\cos\theta) + \frac{1}{2}ml^2\sin^2\theta\,\omega^2$$

The correct effective potential

$$\tilde{v}(\theta) = mgl(1-\cos\theta) - \frac{1}{2}ml^2\sin^2\theta\,\omega^2$$

Plot this as a function of theta

Understand the maxima and minima of the effective potential

$$\frac{\delta\tilde{V}}{\delta\theta} = 0 \rightarrow \frac{maximum}{minimum} = mgl\sin\theta - \frac{1}{2}ml^2\omega^2 2\sin\theta\cos\theta = 0$$

$$\sin\theta\,(gl - l^2\omega^2\cos\theta) = 0$$

Either sin(theta) can vanish or $\frac{g}{l} = \omega^2\cos\theta \rightarrow \cos\theta = \frac{g}{l\omega^2} < 1$

Only possible if

$$\omega^2 > \frac{g}{l}$$

CASE 1: $\omega^2 < g/l$

Potential has 2 extrema at $\theta = 0\ and\ \pi$

CASE 2: $\omega^2 > g/l$

3 extrema

$\theta = 0, \pi$

New extremum at

$$cos\,\theta = \frac{g}{l\omega^2}$$

$\theta = 0 \Rightarrow unstable$

Pendulum constrained to rotate on it's axis

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\,\dot{\phi}^2 - mgl(1-\cos\theta)$$

2 cases to consider

i. When $\dot{\phi}$ is constrained by an exernal torque to be constant, $\boxed{\dot{\phi} = \omega}$
ii. No external forces/torques in the system

Case i. Plug $\dot{\phi} = \omega$ in L:

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\,\omega^2 - mgl(1-\cos\theta)$$

$$\bar{L} = \frac{1}{2}ml^2\dot{\theta}^2 - \tilde{V}$$

$$\boxed{\tilde{V} = effective\ potential = mgl(1-\cos\theta) - \frac{1}{2}ml^2\omega^2\sin^2\theta}$$

What does it look like qualitatively?

Maxima+minima at

$$\frac{d\tilde{V}(\theta)}{d\theta} = 0 \Rightarrow mgl\sin\theta = m\omega^2 l^2 \sin\theta\cos\theta$$

$$\boxed{\sin\theta\left(\frac{g}{\omega^2 l} - \cos\theta\right) = 0}$$

3 possible solutions

$\theta = 0$

$\theta = \pi$

$\cos\theta = \frac{g}{\omega^2 l}$

Only possible if

$\omega^2 > \frac{g}{l}$

Which of these is a maxima or minima

One way to do this is find

$\frac{d^2\tilde{V}}{d\theta^2}$

and check if >0 or <0

$$\frac{d^2\tilde{V}}{d\theta^2} = mgl\cos\theta - m\omega^2 l^2(\cos^2\theta - \sin^2\theta)$$

At $\theta = 0$:

$$\tilde{V}''(\theta = 0) = mgl - m\omega^2 l^2 = m\omega^2 l^2\left(\frac{g}{l\omega^2} - 1\right)$$

At $\theta = \pi$

$$\tilde{V}''(\theta = \pi) = -mgl - m\omega^2 l^2 < 0$$

Easy case is

$\frac{g}{l\omega^2} > 1$

When $\theta = 0\&\pi$ are the only extrema

$\tilde{V}(\theta)$ near an extremum θ_0

$$\tilde{V}(\theta) \sim \tilde{V}(\theta_0) + \tilde{V}(\theta_0)(\theta - \theta_0) + \frac{1}{2}\tilde{V}''(\theta_0)(\theta - \theta_0)^2$$

$f(x) \rightarrow$ look at behaviour near x=a

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)$$

Near $\theta = 0$:

$\tilde{V}(0) = 0$

$\tilde{V}'(0) = 0$

$\tilde{V}''(0) = m\omega^2 l^2\left(\frac{g}{\omega^2 l} - 1\right)$

For $\omega^2 < \frac{g}{l}$, $\tilde{V}''(0)$ is positive

$$\tilde{V}(\theta) \approx \frac{1}{2}\theta^2 * m\omega^2 l^2\left(\frac{g}{\omega^2 l} - 1\right)$$

$m\omega^2 l^2\left(\frac{g}{\omega^2 l} - 1\right)$ *plays the role of k*

$$\boxed{\begin{array}{c}\text{Basic SH oscillations}\\ T = \frac{1}{2}m\dot{x}^2 \\ V = \frac{1}{2}kx^2 \\ \omega = \sqrt{\frac{k}{m}}\end{array}}$$

Frequency of oscillations around $\theta = 0$

$$\omega = \sqrt{\frac{g}{\omega^2 l} - 1}$$

This is different from the usual pendulum

Case ii. $\omega^2 < \frac{g}{l}$
Frequency of oscillations

$$\nu = \sqrt{\frac{g}{l} - \omega^2}$$

Increasing ω^2 beyond g/l
New extremum occurs at an angle θ_0

$$\cos\theta_0 = \frac{g}{\omega^2 l}$$

$$\tilde{V}'(\theta_0)=0$$

Frequency of small oscillations

$$\tilde{V}''(\theta_0) = -\frac{mg^2}{\omega^2} + m\omega^2 l^2 > 0$$

$$\nu = \sqrt{\frac{\tilde{V}''(\theta_0)}{ml^2}} = \sqrt{\omega^2 - \frac{g^2}{l^2\omega^2}}$$

Case II: no external constraint or torque
Angular momentum conserved

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\,\dot{\phi} - mgl(1-\cos\theta)$$

CAN'T PLUG IN $\dot{\phi} = \omega$

$$P_\theta = \frac{\delta L}{\delta\dot{\theta}} = ml^2\dot{\theta}$$

$$P_\theta = \frac{\delta L}{\delta\phi} = ml^2\sin^2\theta\,\dot{\phi}$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\phi}}\right) = \frac{\delta L}{\delta\phi} = 0$$

P_ϕ conserved

$$ml^2\sin^2\theta\,\dot{\phi} = P_\phi$$

$$\dot{\phi} = P_\theta/ml^2\sin^2\theta$$

Eqn for θ

$$ml^2\ddot{\theta} = \frac{P_\phi^2}{ml^2\sin^3\theta}\cos\theta - mgl\sin\theta$$

$$\tilde{V} = mgl(1-\cos\theta) + P_\phi^2/2ml^2\sin^2\theta$$

Mgl=constant, P_ϕ^2=constant

If we write

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - \tilde{V}$$

$$\frac{d\tilde{V}}{d\theta} = 0$$

Gives the minimum

$$mgl\sin\theta - \frac{P_\phi^2\cos\theta}{ml^2\sin^3\theta} = 0$$

Kepler problem

31 March 2011
14:38

The gravitational potential is
Newton:

$$V = -\frac{Gm_1 m_2}{|\overline{r_1} - \overline{r_2}|}$$

Step I: choose "good" coords.

i) $\bar{r} = \overline{r_2} - \overline{r_1}$ (relative position vector)

ii) $\bar{R} = \frac{[m_1 \overline{r_1} + m_2 \overline{r_2}]}{[m_1 + m_2]}$

Location of centre of mass

iii) If no external forces, then $\bar{R} = 0$

Step II: write Ke& Pe

$$T = \frac{1}{2} m_1 \dot{\overline{r_1}}^2 + \frac{1}{2} m_2 \dot{\overline{r_2}}^2$$

Plug in for $\overline{r_1}$& $\overline{r_2}$

$$\overline{r_1} = \bar{R} - \frac{m_2}{m_1 + m_2} \bar{r}$$

$$\overline{r_2} = \bar{R} + \frac{m_2}{(m_1 + m_2)} \bar{r}$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{\bar{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\bar{r}}^2$$

$$\frac{1}{2}(m_1 + m_2)\dot{\bar{R}}^2 \rightarrow centre\ of\ mass\ kinetic\ energy$$

$$\frac{m_1 m_2}{m_1 + m_2} \rightarrow reduced\ mass$$

$$V = -\frac{Gm_1 m_2}{r}$$

Potential energy does NOT depend on $\bar{R}$

Translational symmetry for the centre of mass

Equation of motion of $\bar{R} \Rightarrow$ conservation of linear momentum

(exercise)

Interesting part

$$L = \frac{1}{2}\mu \dot{\bar{r}}^2 - V(r)$$

$$V(r) = -\frac{Gm_1 m_2}{r}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\dot{\bar{r}}^2 = [velocity]^2$$

Exactly equivalent to problem of particle in "central potential"

Natural to go to polar coordinates

$$L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 - V(r)$$

$$\dot{r}^2 = radial\ velcoity^2$$

$$r^2 \dot{\theta}^2 = tangential\ velocity^2$$

01 April 2011
09:15

$$\omega = \sqrt{\frac{k}{m}}$$

$$m\ddot{x} = -kx$$

Special Relativity

05 May 2011
09:13

Inertial reference frame-> observer is moving with constant velocity (no acceleration)
    Observer carrying clocks and rulers
Non-inertial reference frame: reference frame subject to acceleration
    Trajectory of object observed from rotating frame->fictitious force appears to act on ball (pseudo-force)
        Coriolis "force"
(*)Inertial frame: laws of physics are the same in all inertial reference frames
    Frames moving with respect to each other with velocity v
    How is (x', y', z') coordinates system related to (x, y, z) coordinate systemn
    Galilean transformations

$$\boxed{\begin{array}{c} y' = y - v_y t \\ z' = z - v_z t \\ x' = x - v_x t \\ t' = t \end{array}}$$

Newton's law in first frame

$$\bar{F} = m\ddot{\bar{x}} = m\frac{d^2}{dt^2}\bar{r} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k})$$

$$y' = y - v_y t$$

$$v_y = constant$$

$$\frac{d}{dt}(y') = \frac{dy}{dt} - v_y$$

$$\boxed{\frac{d^2}{dt^2}(y') = \frac{d^2y}{dt^2}}$$

$$\bar{F} = m(\ddot{x}'\hat{\imath} + \ddot{y}'\hat{\jmath} + \ddot{z}'\hat{k})$$

Einstein (*) speed of light in vacuum is the same for all inertial observers
Special relativity
    Gedanken experiments
    Time dilation effect
        Observer A: in box moving with velocity v. light on ceiling at height L
        Observer B (at "rest")

    Observer A: time taken for photon to hit ground

$$t = \frac{L}{c}$$

    Observer B: time taken

$$t' = \frac{\sqrt{L^2 + v^2 t'^2}}{c}$$

        Solve for t'

$$(t')^2 = \frac{L^2 + v^2 (t')^2}{c}$$

$$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Muon: μ^- → unstable →half-life at rest $2 \times 10^{-6} s$
    Moving muon has larger lifetime as predicted by time dilation
    For muon moving at 0.6c

$$\frac{1}{\sqrt{1-0.6^2}} = \frac{5}{4}$$

Lifetime= $\frac{5}{4} \times 2 \times 10^{-6} s$

Loss of simultaneity

Length contraction

$$Length\ L$$
$$source - \quad - \quad - \quad - \quad ------- \quad light$$
$$detector - \quad - \quad - \quad - \quad ------- \quad reflected \quad \rightarrow v$$
$$observer A$$

Observer B

For observer A

Time taken for photon to travel from left to right and back

$$\frac{2l}{c} = t$$

For observer B

Let T1 be time taken to get to the right side

$$ct_1 = l' + vt_1$$

$$t_1 = \frac{l'}{c-v}$$

T2=time to get back

$$t_2 = \frac{l' - vt_2}{c}$$

$$t_2 = \frac{l'}{c+v}$$

$$t' = t_1 + t_2 = \frac{l'}{c-v} + \frac{l'}{c+v}$$

$$t' = \frac{2l'c}{c^2 - v^2}$$

From time dilation formula

$$t' = \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{2l'c}{c^2 - v^2} = \frac{2l}{c\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{2l'}{1-\frac{v^2}{c^2}} = \frac{2l}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{l'}{\sqrt{1-\frac{v^2}{c^2}}} = l$$

$$\boxed{l' = \sqrt{1-\frac{v^2}{c^2}}\, l}$$

Potential paradox

How is time dilation reconciled with relativity