Euler-Lagrange

$$I = \int_a^b f(g, g')dx$$

Is extremised when

$$\frac{\delta f}{\delta g} = \frac{d}{dx}\left(\frac{\delta f}{\delta g'}\right)$$

Examples:

a) Soap bubble- surface tension acts to minimise surface area
b) Hamilton's principle in classical mechanics in 1d example we recovered f=ma

Generalisations

1) Integrals in more than 1 dimension e.g.

$$I = \int f\left(g(\underline{x}), \frac{\delta g}{\delta x_1}, \frac{\delta g}{\delta x_2}, \dots\right) dx_1, dx_2 \dots$$

$$\underline{x} = (x_1, x_2, \dots)$$

Euler-Lagrange equation

$$\frac{\delta f}{\delta g} = \frac{\delta}{\delta x_i}\left(\frac{\delta f}{\delta\left(\frac{\delta g}{\delta x_i}\right)}\right)$$

Summation over repeated index is implied

2) If there are more fields, g, e.g. $f(g, h, g', h')$, we have Euler Lagrange equations for each one.

If the fields are $g_1, g_2, g_3 \dots g_m$

We have

$$\frac{\delta f}{\delta g_a} = \frac{\delta}{\delta x_i}\left(\frac{\delta f}{\delta\left(\frac{\delta g_a}{\delta x_i}\right)}\right)$$

This is m equations for a=1, .. , m

Example:

A real particle physics Lagrangian

$$L = \left(\frac{\delta \phi}{\delta x_i}\right)\left(\frac{\delta \phi}{\delta x^i}\right) - m^2\phi^2$$

i=0,1,2,3

$$x_i = g_{ij}x^j$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We will get our equations of motion by extremising

$$\int L\, d^4x$$

i.e. integrate over whole space-time

Now just put this into Euler-Lagrange equation:

$$\frac{\delta L}{\delta \phi} = \frac{\delta}{\delta x_i}\left(\frac{\delta L}{\delta\left(\frac{\delta \phi}{\delta x_i}\right)}\right)$$

$$\frac{\delta L}{\delta \phi} = -2m^2\phi$$

$$\frac{\delta L}{\delta\left(\frac{\delta \phi}{\delta x_i}\right)} = 2\frac{\delta \phi}{\delta x^i}$$

-write it out long hand, or think of

$$\frac{\delta \phi}{\delta x_i}\frac{\delta \phi}{\delta x^i} = g^{ij}\frac{\delta \phi}{\delta x_i}\frac{\delta \phi}{\delta x_j}$$

So Euler-Lagrange

$$\Rightarrow -2m^2\phi = \frac{\delta}{\delta x_i}\left(2\frac{\delta\phi}{\delta x^i}\right)$$

$$\Rightarrow -m^2\phi = \left(\frac{\delta}{\delta x_0}\frac{\delta}{\delta x^0} + \frac{\delta}{\delta x_1}\frac{\delta}{\delta x^1} + \frac{\delta}{\delta x_2}\frac{\delta}{\delta x^2} + \frac{\delta}{\delta x_3}\frac{\delta}{\delta x^3}\right)\phi$$

$$= \left(\frac{\delta^2}{\delta t^2} - \frac{\delta^2}{\delta x^2} - \frac{\delta^2}{\delta y^2} - \frac{\delta^2}{\delta z^2}\right)$$

$$\Rightarrow \boxed{\left(\frac{\delta^2}{\delta t^2} - \nabla^2 + m\right)\phi = 0}$$

(*)

This is the Klein-Gordon equation

Consider plane-wave solutions

$$\phi = e^{i\underline{k}.\underline{x} - i\omega t}$$

Subbing into (*),

$$[-\omega^2 + \underline{k}.\underline{k} + m^2]\phi = 0$$

For this to be a solution we need $\omega^2 = |\underline{k}|^2 + m^2$

c.f. $E^2 = p^2 + m^2$

The klein-gordon equation describes relativistic scalar particles of mass m.

Notes

i) $L = \left(\frac{\delta\phi}{\delta x_i}\right)\left(\frac{\delta\phi}{\delta x^i}\right) - m^2\phi^2$

Looks (a bit) like KE-PE

ii) $m^2\phi^2$ term gives the mass to the particle

Differential Equations

3) Series solutions (Frobenius' method)

Consider $y'' + P(x)y' + Q(x)y = 0$

(**)

[linear, 2nd order, homogeneous (no RHS)]

Some definitions

If as $x \to x_0$: P, Q finite $\to$ x_0 regular point

If $(x - x_0)P, (x - x_0)^2 Q$ finite $\to$ x_0 is regular singular point

If $(x - x_0)P$ or $(x - x_0)^2 Q$ diverge $\to$ x_0 is irregular singular point

Fuch's theorem

If x_0 is regular or regular singular there exists at least one power series solution to (**)

Method

Assume a solution of the form

$$y = \sum_{n=0}^{\alpha} C_n (x - x_0)^{n+\sigma}$$

$$c_0 \neq 0$$

Sub into (x) and expand. Then set coefficients to ensure each power of $x - x_0$ vanishes

Example. consider a power series solution to $y'' + y = 0$ about $x = 0$

Set

$$y = \sum_{n=0}^{\alpha} c_n x^{n+\sigma}$$

And sub in

$$\sum_{n=0}^{\alpha} C_n\left((n+\sigma)(n+\sigma-1)x^{n+\sigma-2} + x^{n+\sigma}\right) = 0$$

Write out sum

$$c_0\sigma(\sigma-1)x^{\sigma-2} + c_1(\sigma+1)(\sigma)x^{\sigma-1} + c_2(\sigma+2)(\sigma+1)x^{\sigma} + c_0 x^{\sigma} + x^{n+\sigma}c_{n+2}(n+\sigma+2)(n+\sigma+1) + c_n x^{n+\sigma} = 0$$

For the whole thing to vanish, we need the coefficients of each power of x to vanish.

$\Rightarrow c_0\sigma(\sigma-1)=0 \Rightarrow \sigma=0, \text{or } \sigma=1 \text{ as } c_0\neq 0$
$c_1(\sigma+1)\sigma=0$
$c_{n+2}(n+\sigma+2)(n+\sigma+1)+c_n=0$

i) Indicial equation: Look at the lowest power of x. Remember $c_0\neq 0 \Rightarrow \sigma(\sigma-1)=0 \text{ or } \sigma=0$
ii) Next order: We need $c_1(\sigma+1)\sigma=0$

Options:

If $\sigma=1\Rightarrow C_1=0$
If $\sigma=0\Rightarrow C_1$ is not determined

-Corresponds to mixing in an arbitrary amount of the other sol'n

In order to separate the two solutions, we consider $c_1=0$

iii) General power

$$c_{n+2}(n+2+\sigma)(n+1+\sigma)+c_n=0$$
$$\Rightarrow c_{n+2}=-\frac{c_n}{(n+2+\sigma)(n+1+\sigma)}$$

Two solutions

1) $\sigma=0$

$$c_{n+2}=-\frac{c_n}{(n+1)(n+2)}; y=c_0\left\{1-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right\}$$
$$=c_0\cos(x)$$

2) $\sigma=1$

$$c_{n+2}=-\frac{c_n}{(n+2)(n+3)}:\; y=c_0\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right\}$$
$$=c_0\sin(x)$$

Comments

In this case, both solutions converge for all x.
(*) we have two power law solutions

Example 2.

$$y''+\frac{y'}{r}+\left(k^2-\frac{\nu^2}{r^2}\right)y=0 \text{ about } x=9$$

This is Bessel's equation- arises whenever you work in cylindrical polars. Recalling the definitions r=0 is a regular singular point ⇒ we expect one series solution

Try

$$y=\sum_{n=0}^{\alpha} c_n r^{n+\sigma}$$

Bang the guess into the equation

$$\Rightarrow \sum_{n=0}^{\alpha} c_n\{(n+\sigma)(n+\sigma-1)r^{n+\sigma-2}+(n+\sigma)r^{n+\sigma-2}-\nu^2 r^{n+\sigma-2}+k^2r^{n+\sigma}\}=0$$

Rearrange in terms of powers of r

$$\Rightarrow r^{\sigma-2}(r^2-\nu^2)c_0+r^{\sigma-1}\left((\sigma+1)^2-\nu^2\right)c_1+r^{\sigma}\left(c_2\left((\sigma+2)^2-\nu^2\right)+c_0k^2\right)+\cdots$$
$$+r^{n+\sigma}(c_{n+2}[(\sigma+n+2)^2-\nu^2]+c_nk^2)+\cdots$$

Again work power by power.

Indicial equation: $\sigma^2=\nu^2\Rightarrow \sigma=\pm\nu$

$r^{\sigma-1}$ term: $c_1\left((\sigma+1)^2-\nu^2\right)=0\Rightarrow c_1=0$ in general

General term: recurence relation

$$c_{n+2}[(\sigma+n+2)^2-\nu^2]+c_nk^2=0\Rightarrow c_{n+2}=\frac{k^2c_n}{(\sigma+n+2)^2-\nu^2}$$

Solutions

$$y=r^{\nu}\left\{1-\frac{k^2r^2}{2(2+2\nu)}+\frac{k^4r^4}{\cdots}+\cdots\right\}$$
$$y=r^{-\nu}\left\{1-\frac{k^2r^2}{2(2-2\nu)}+\frac{k^4r^4}{2(2-\nu)+(4-2\nu)}+\cdots\right\}$$

In the $\sigma=-\nu$ case, the determinator factor in the recurence relation is

$$(n+2-\nu)^2-\nu^2=(n+2)(n+2-2\nu)$$

As n is even in our expansion, this vanishes if ν is an integer
$\Rightarrow$ if ν is an integer the coefficients of the $\sigma = \nu$ solution blow up$\Rightarrow$ bad!
For ν =integer, we only have one solution
For $\nu \neq$ integer at large n we have

$$c_{n+2} \sim -\frac{c_n}{n^2}$$

$\Rightarrow$convergent for all r.

A second solution

In situations (such as ν =integer case above) where we have only 1 soution$\Rightarrow$ 2nd soluton doesn't have a power series expansion about r=0
$\Rightarrow$try a function of the form

$$\mathrm{y} - \ln x \left(\sum c_n x^{n+\sigma} \right) + \sum d_n x^{n+\sigma}$$

Subbing in to our equation gives 2 types of term. If the $\ln(x)(\sum c_n x^{n+\sigma})$ isnt hit by a derivative, we have $\ln(x)$ where our equation is $\hat{L}y = 0$
If a derivative hits the h(x) we have powers of x
The other class of term is

$$\tilde{\sum_n} c_n x^{n+\sigma} + \hat{L}\left(\sum (d_n x^{n+\sigma})\right)$$

-coefficients are determined by c_n

If we are to have solution, both sets of terms must vanish independently
If $\ln(x)\, \hat{L}(\sum c_n x^{n+\sigma})$ is to vanish. We must set $\sum c_n x^{n+\sigma}$ to be the original power series solution
The remaining power series terms are then determined order by order as before

2nd solution = ln(x)(first solution)+(different power series)