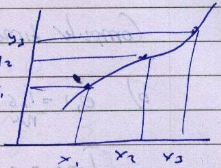
No 7 from theory.

Past paper 2010

**A1**

It is of the form $y(x) = y_1 \cdot O + y_2 \cdot O + y_3 \cdot O$




we want $y(x_1) = y_1$

$y(x_2) = y_2$

$y(x_3) = y_3$

$$y(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} y_3$$

Our points are $(1,1)$, $(2,5)$, $(3,6)$

$$y(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} \cdot 1 + \frac{5(x-1)(x-3)}{(2-1)(2-3)} + 6\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

$$= \frac{1}{2}(x-2)(x-3) - 5(x-1)(x-3) + 3(x-1)(x-2)$$

$a = \frac{1}{2}, \; b = 1, \; c = 3$

**A2**

i) let $I = 0$, $N = 0$

ii) Select randomly a number between $-R$ and $+R$

iii) must satisfy $x^2 + y^2 \le R$ then a) $I = I + h(x,y)$

b) $N = N + 1$

iv) Repeating $N \simeq 100$

$Vol = Area \times height = \pi R^2 \cdot \frac{I}{N}$

$\boxed{A3}$

Error $= \dfrac{10}{N^2}$

Require error $: 0{,}1 = \dfrac{10}{N^2} \Rightarrow \underline{\underline{N = 10}}$

Compute time $= 0{,}2s \times N = 2\,\text{sec}$

b) $0{,}1 = \dfrac{1{,}6}{N^4} \rightarrow N^4 = \dfrac{1{,}6}{0{,}1} \Rightarrow N^4 = 16 \Rightarrow \underline{\underline{N = 2}}$

Time $= 0{,}2s$

$\boxed{A4}$

i) Euler is only exact for straight lines so only a) would be correct. $N=0$

ii) Midpoint section works for parabola $N=0$ and $N=1$ so a), b) ~~and~~

iii) The runge-kutta $\rightarrow N=3$ so all of them would be satisfied with a runge kutta.

| | Euler | Midpoint | R.K |
|---|---|---|---|
| a | ✓ | ✓ | ✓ |
| b | × | ✓ | ✓ |
| c | × | × | ✓ |
| d | × | × | × |

↳ d is an exponential! not a polynomial

B1

a) $V = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{y_i - \bar{y}}{e_i} \right)^2$

b) Consider $V = V(\bar{y})$

requiring minimum variance

$$\frac{\partial V}{\partial \bar{y}} = 0 \Rightarrow \frac{1}{N} 2 \sum \frac{(y_i - \bar{y})}{e_i^2} = 0$$

$$\Rightarrow \sum \frac{y_i}{e_i^2} = \bar{y} \sum \frac{1}{e_i^2} \Rightarrow \bar{y} = \frac{\sum \frac{y_i}{e_i^2}}{\sum \frac{1}{e_i^2}}$$

c) bookwork.

↳ Statistics and Curve fitting

§4.5 p. 8.

d) This time, we use the function $f(x) = ax^3 + b$

$$\chi^2 = \sum \frac{(ax^3 + b - y_i)^2}{e_i^2}$$

We want $\frac{\partial \chi^2}{\partial a} = 0$ and $\frac{\partial \chi^2}{\partial b} = 0$

i) $2 \sum \frac{(ax^3 + b - y_i)(x^3)}{e_i^2} = 0 \Rightarrow a \sum \frac{x^3 \cdot x^3}{e_i^2} + b \sum \frac{x^3}{e_i^2} - \sum \frac{y_i x^3}{e_i^2} = 0$

ii) $2 \sum \frac{(ax^3 + b - y_i)}{e_i^2} 1 = 0 \rightarrow a \sum \frac{x^3}{e_i^2} + b \sum \frac{1}{e_i^2} - \sum \frac{y_i}{e_i^2} = 0$

B2

Consider $f(x) = 0$

a) describe iterative function.

a) find $a, b$, such that $\text{sign}(f(a)) = -\text{sign}(f(b))$

define $x = \frac{(a+b)}{2}$

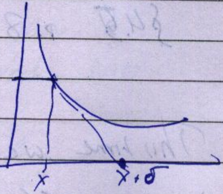
if $\text{sign}\, x = \text{sign}\, f(a)$

then $x = a$

if . . . . . $\text{sign}\, f(b) \rightarrow x = b$

Repeat N times.

b)




so $\text{grad} = f'(x) = \frac{-f(x)}{\delta}$

$$\boxed{\delta = -\frac{f(x)}{f'(x)}}$$

Algorithm: start with $x$, increment with $x + \delta$ N times.

c) $f(x) = cx^2 + bx + c = 0$

$f'(x) = 2cx + b$

$\delta = -\frac{ax^2 + bx + c}{2cx + b}$, $x_0 = 0$ $\delta = -\frac{c}{b}$

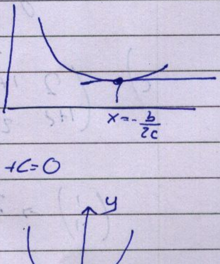
$x_0 = 0 \Rightarrow x_1 = x_0 + \delta(x_0) = -\frac{c}{b}$

$x_2 = x_1 + \delta(x_1) = -\frac{c}{b} + \frac{\left(-a\left(\frac{c}{b}\right)^2 + b\left(\frac{c}{b}\right) - c\right)}{\left(2c\left(-\frac{c}{b}\right) + b\right)}$

$$\boxed{x_2 = -\frac{c}{b} - \frac{ac^2}{-2cbc + b^3}}$$

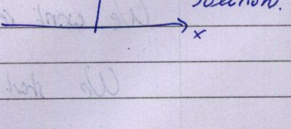
Fails when $f'(x) = 0$, so $2cx + b = 0$ $x = -\frac{b}{2c}$

this method fails at this point.




$b^2 - 4ac < 0$, no real solutions for $cx^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$




if no solutions and N.R method was used, then it would find the minimum and then diverge.

What would separate you from the "history" student is for you to be able to find a solution to a new problem that wasn't in your book. Need to be able to use your brain.

$\boxed{B3}$

a) in book

b) $A\vec{x} = \lambda\vec{x}$

$\times(-1)$, $A(-\vec{x}) = -\lambda\vec{x} = \lambda(-x)$ still an eigenvector
it doesn't matter if

opposite direction

let $\bar{y} = \bar{x}$ $Ay = \lambda y$
$y = c_i$

c) $\begin{pmatrix} 2 & 1+\varepsilon \\ 1+\varepsilon & 2 \end{pmatrix}$ $\varepsilon \ll 1$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \to \lambda = 1$ , $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \lambda = 3$

We want results very close to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

We start where the last known eigenvectors.

$\vec{Y} : \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $AY = \begin{pmatrix} 2+1+\varepsilon \\ 1+\varepsilon+2 \end{pmatrix} = \begin{pmatrix} 3+\varepsilon \\ 3+\varepsilon \end{pmatrix}$

$A^n Y = \begin{pmatrix} 2 & 1+\varepsilon \\ 1+\varepsilon & 2 \end{pmatrix}\begin{pmatrix} 3+\varepsilon \\ 3+\varepsilon \end{pmatrix} = \begin{pmatrix} 6+3+2\varepsilon+3\varepsilon+\varepsilon+O(\varepsilon^2) \\ 3+6+3\varepsilon+2\varepsilon+\varepsilon+O(\varepsilon^2) \end{pmatrix}$

$$A^T \vec{y} = \begin{pmatrix} 9+6\varepsilon \\ 9+6\varepsilon \end{pmatrix} + O\varepsilon^2$$

So "$\frac{2}{1}$", $\lambda = \dfrac{9+6\varepsilon}{3+\varepsilon} = \dfrac{3+2\varepsilon}{1+\frac{1}{3}\varepsilon} = (3+2\varepsilon)(1-\frac{1}{3}\varepsilon)$

$$= 3+2\varepsilon - \varepsilon$$

$$= \boxed{3+\varepsilon}$$

Checking result

$$\begin{vmatrix} 2-\lambda & 1+\varepsilon \\ 1+\varepsilon & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - (1+\varepsilon)^2 = 0$$

$$\lambda^2 - 4\lambda + 4 - (1+\varepsilon)^2 = 0$$

quadratic $\underline{\lambda = 3+\varepsilon,\ 1-\varepsilon}$

# Very Similar to Exam!

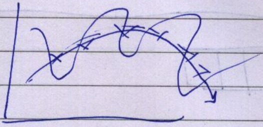
$\boxed{A1}$ Similar to other paper. Bookwork

$\boxed{A2}$ i) $g(x) = a_0 + a_1 x + \dots x^M a_M$

$g$ has $M+1$ coefficients

$\rightarrow \underline{N = M+1}$

ii)



Wild fluctuations a long way from newly added point.

$\rightsquigarrow$ Require local algorithm.

$\boxed{A3}$ -> pipeline

$\boxed{A4}$ $\frac{\partial \phi(x)}{\partial x} = \frac{\phi(x+h) - \phi(x-h)}{2h}$

$$\frac{\partial^2 \phi(x)}{\partial x^2} = \frac{\frac{\partial \phi}{\partial x}\Big|_{x+h} - \frac{\partial \phi}{\partial x}\Big|_{x-h}}{2h}$$

$$= \frac{1}{4h^2}\left(\phi(x+2h) - \phi(x) - (\phi(x) - \phi(x-2h))\right)$$

$$= \frac{1}{4h^2}\left(\phi(x+2h) - 2\phi(x) + \phi(x-2h)\right) \quad 2h \rightarrow h$$

$$= \frac{1}{h^2}\left(\phi(x+2h) - 2\phi(x) + \phi(x-2h)\right)$$

Let $\phi_i = \phi(x)$

Using (*) $\frac{\partial^2 \phi}{\partial x^2} = 0$

become $\phi_{i+1} - 2\phi_i + \phi_{i-1} = 0$

$$\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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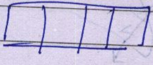
A4



No of possibilities $= 2^4 = 16$

Including 0 → largest number is 15

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B1 a) bookwork

b) $f(x) = x^2 - \frac{1}{3}$ in the $x \in [0,1]$

$f(0) = -\frac{1}{3}$, $f(1) = \frac{2}{3}$, $x = \frac{1}{2}$

$f(\frac{1}{2}) = \frac{1}{4} - \frac{1}{3} = \frac{-1}{12}$ $\quad a = -\frac{1}{12}$ so

$f(\frac{3}{4}) > 0$ $\quad a = \frac{1}{2}$, $b = 3/4$

good enough answer

The ~~rabo~~ solution is between $\frac{1}{2}$ and $\frac{3}{4}$

$= \frac{5}{8} \pm \frac{1}{8}$

c) $\delta = -\frac{f(x)}{f'(x)}$

d) Given i) $F(x) = 3\cos(x)$ , $F' = -3\sin(x)$

$F' = 0 \Rightarrow x = k\pi \quad k \in \mathbb{Z}$

ii) $G(x) = 3x^2 - x$ $\quad G'(x) = 0 \Rightarrow 6x - 1 = 0 \Rightarrow x = \frac{1}{6}$

iii) $H(x) = e^{2x} - 1$ , $H'(x) = 2e^{x} = 0$ $x \to -\infty$

never fail.

B2

a) $\frac{dy}{dx} = f(x, y)$

$y_{n+1} = y_n + f(x_n, y_n) h$

$\frac{d^2y}{dx^2} = 2x \frac{dy}{dx} + 3x^2$ (*)

let $z_1 = y$
$z_2 = \frac{dy}{dx}$ (**)

(*) $\to \frac{dz_2}{dx} = 2x z_2 + 3x^2$ (2)

(**) $\to \frac{dz_1}{dx} = +z_2$ (1)

2 diff. equations.

c) $x=0,\ y=0,\ \frac{dy}{dx}=1$

$\downarrow z_1=0 \qquad \downarrow z_2=1$

From 1st iteration $x=0 \rightarrow x=h$

① $z_1(h)=z_1(0)+\frac{\partial z_1}{\partial x}\Big|_0 \cdot h = 0+z_2(0)\cdot h = h = 0{,}5$

② $z_2(h)=z_2(0)+\cancel{\frac{\partial z_2}{\partial x}}\Big|_0 \cdot h = 1$

2nd iteration

$x=h \rightarrow x=2h=2\cdot 0{,}5$

$=0{,}5 \rightarrow =1$

① $z_1(2h)=z_1(h)+\frac{\partial z_1}{\partial x}\Big|_h \cdot h$

$=0{,}5+z_2(h)\cdot h$

$=0{,}5+1\cdot h=1$

② $z_2(2h)=z_2(h)+\frac{\partial z_2}{\partial x}\Big|_h \cdot h$

$=1+(2h+3h^2)\cdot h = 15/8$

B3

a) Bookwork

any vector can be written as a lin comb $\Leftrightarrow$ basis

b) Show that $A \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

has eigenvectors $\vec{e_1}, \vec{e_2}, \vec{e_3}$ with $\lambda_1, \lambda_2, \lambda_3$

$$A\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}4\\4\\0\end{pmatrix} = 4\begin{pmatrix}1\\1\\0\end{pmatrix} \to \lambda_1 = 4$$

$$A\begin{pmatrix}1\\-1\\0\end{pmatrix} = \dots \quad 2\begin{pmatrix}1\\-1\\0\end{pmatrix} = \lambda_2 = 2$$

$$A\begin{pmatrix}0\\0\\1\end{pmatrix} = \dots \quad 1\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \lambda_3 = 1$$

c) same as previous paper

d) The best choice is $\begin{pmatrix}0\\0\\1\end{pmatrix}$ with $\lambda = 1$

e)
$$A'\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix} 3 & 1+\varepsilon & 0 \\ 1+\varepsilon & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}4+\varepsilon\\4+\varepsilon\\0\end{pmatrix}$$

$$A\begin{pmatrix}4+\varepsilon\\4+\varepsilon\\0\end{pmatrix} = \begin{pmatrix}16+8\varepsilon\\16+8\varepsilon\\0\end{pmatrix} + \mathcal{O}(\varepsilon^2)$$

$$\frac{A(Y)}{Y} = \frac{4+\varepsilon}{1} = \underline{4+\varepsilon}$$

$$\frac{(A^2Y)}{(AY)} = \frac{16+8\varepsilon}{4+\varepsilon} = \boxed{\frac{4(4+2\varepsilon)}{4(1+\frac{1}{4}\varepsilon)}} = (4+2\varepsilon)(1-\tfrac{1}{4}\varepsilon) + \mathcal{O}(\varepsilon^2)$$

$$= 4+\varepsilon+\mathcal{O}(\varepsilon^2)$$

$\boxed{B3}$ a)

$$\lambda_{max} = \frac{\left(A^{n+1}\vec{Y}\right)_j}{\left(A^{n}\underline{Y}\right)_j}$$

and go through derivation.
Sketch the derivation.